Continuous Time Markov Chains

As with the PP, we first define what we would like a CTMC to be, and then figure out if we can construct it...

· <u>Defn</u> (Continuous Time Markov Chain) Let $\mathcal{X}$ be a countable set (the state-space). A $\mathcal{X}$-valued stochastic process $X = \{X(t)\}_{t \in \mathbb{R}_+}$ is called a CTMC if $\forall t, s \geq 0$, and $\forall k \in \mathbb{N}$, $0 \leq t_1 < t_2 < \dots < t_k < t$, and $\forall x_1, x_2, \dots, x_k, x, y \in \mathcal{X}^{k+2}$,

$$\mathbb{P}[X(t+s) = y \mid X(t) = x, X(t_k) = x_k, \dots X(t_1) = x_1] = \mathbb{P}[X(t+s) = y \mid X(t) = x]$$

Moreover, the chain is said to be homogeneous if the RHS is independent of t, i.e., $\mathbb{P}[X(t+s) = y \mid X(t) = x] = P_s(x,y)$

Before proceeding, some comments:

1) Does a CTMC exist? Well, we already saw one! Check that if $N(t)$ is a PP(λ), then $N(t)$ is a CTMC on $\mathbb{N}$.
In particular, $P_s(x,y) = \dfrac{e^{-\lambda s} (\lambda s)^{y-x}}{(y-x)!}$ $\forall y \geq x$, and 0 o.w.

2) We use the notation $P_s(x,y)$ to maintain a similarity to the notation for DTMCs, where we wrote $P(x,y)$ for the transition probabilities. Note though that P_s is not one matrix but a fn of s, and that $P_s \neq P^s$ (unlike in a DTMC)

3) As with PP, we will see 2 main ways to construct a CTMC
   i) 'Analytic' - Via the 'transition semigroup'
   ii) 'Probabilistic' - Via an embedded discrete-time 'jump chain'

CTMCs via the Transition Semigroup

Given the above defn of a CTMC, let $P(t) = \{P_t(x,y)\}_{x,y\in\mathcal{X}}$, (where $P_t(x,y) \triangleq \mathbb{P}[X(s+t)=y \mid X(s)=x] \;\forall s,t \geq 0$)

As with DTMCs, such a P must satisfy consistency eqns:

- (Chapman-Kolmogorov Eqns) $\forall t,s \geq 0$, and $\forall x,y\in\mathcal{X}$, we have $P_{t+s}(x,y) = \sum_{z\in\mathcal{X}} P_t(x,z)P_s(z,y)$, or compactly

$$P_{t+s} = P_t P_s \;,\; P_0 = I$$

- Moreover, let $\pi(t) = \{\pi_x(t)\}_{x\in\mathcal{X}}$ be the distribution of $X(t)$. Then we have $\forall t \geq 0$, $\pi(t)^T = \pi(0)^T P_t$

- The problem is that there is no particular t s.t P_t can be used to determine P_s for any $s \neq t$ (unlike in a DTMC, where $P_1 = P$ and $P_n = P^n \;\forall n\in\mathbb{N}$). Instead we need to define P_t in terms of an 'infinitesimal generator'. We outline this first for some simple cases.

Eg - (Poisson Process) First consider $N(t) \sim PP(\lambda)$. By defn, we have

$$P_t(x,y) = \mathbb{P}[N(t+s)=y \mid N(s)=x] \;\forall t,s\geq 0$$

$$= \frac{e^{-\lambda t}(\lambda t)^{y-x}}{(y-x)!}\mathbb{1}_{\{y\geq x\}}$$

- On the other hand, recall we defined a $PP(\lambda)$ via the eqns $\mathbb{P}[N(t+\delta)-N(t)=1] = \lambda\delta + O(\delta^2)$, $\mathbb{P}[N(t+\delta)-N(t)=0] = 1-\lambda\delta+O(\delta^2)$ and $\mathbb{P}[N(t+\delta)-N(t)>1] = O(\delta^2)$

Using this, we can write

$$\mathbb{P}[N(t+\delta)=y] = \sum_x \mathbb{P}[N(t)=x]\,\mathbb{P}[N(t+h)=y \mid N(t)=x]$$
$$= \lambda\delta\,\mathbb{P}[N(t)=y-1] + (1-\lambda\delta)\,\mathbb{P}[N(t)=y] + O(\delta^2)$$

- Let $\pi_t(x) = \mathbb{P}[N(t)=x]$. Then we can write

$$\pi_{t+\delta}(x) - \pi_t(x) = \pi_t(x-1)\lambda\delta - \pi_t(x)\lambda\delta + O(\delta^2)$$
$$\Rightarrow \frac{\pi_{t+\delta}(x) - \pi_t(x)}{\delta} = \lambda(\pi_t(x-1) - \pi_t(x)) + O(\delta)$$

Taking limit $\delta \searrow 0$, we get the differential eqns

$$\pi_t'(x) = \lambda(\pi_t(x-1) - \pi_t(x)) \ \forall x \geq 1, \ \pi_t'(0) = -\lambda\pi_t(0), \ \pi_0(x) = \mathbb{1}_{\{x=0\}}$$

- One way to solve these is to first solve for $\pi_t(0)$ as

$$\int_1^{\pi_0(t)} \frac{d\pi_t(0)}{\pi_t(0)} = \int_0^t -\lambda\, dt \Rightarrow \pi_t(0) = e^{-\lambda t}$$

- Next, for $\frac{d\pi_t(1)}{dt} = \lambda e^{-\lambda t} - \lambda\pi_t(1)$, we can write it as

$$e^{\lambda t}\frac{d\pi_t(1)}{dt} + \lambda e^{\lambda t}\pi_t(1) = \frac{d}{dt}\left(e^{\lambda t}\pi_t(1)\right) = \lambda, \text{ and } \pi_0(1) = 0$$

Solving we get $\pi_t(1) = \lambda t e^{-\lambda t}$. Moreover, we can continue this via induction to show $\pi_t(x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$

- The differential eqns can be written concisely as $d\pi_t/dt = \pi_t^T Q$, where

$$Q(x,y) = \begin{cases} \lambda & ; y = x+1 \\ -\lambda & ; y = x \\ 0 & ; \text{ow} \end{cases} = \begin{pmatrix} -\lambda & \lambda & 0 & \cdots \\ 0 & -\lambda & \lambda & \cdots \\ & 0 & -\lambda & \lambda & \cdots \\ \vdots & \vdots & & \ddots \end{pmatrix}.$$

The system has a unique solution $\pi_t(x) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}$ if $\pi_0(x) = \mathbb{1}_{\{x=0\}}$.

Eg - (Flip-Flop chain) Let $N(t)$ be the Poisson process, and define $X(t) \in \{-1,1\}$ as $X(t) = X(0)(-1)^{N(t)}$, $X(0) \sim U$ on $\{-1,1\}$.

Now
$$P_t(1,1) = \mathbb{P}[X(s+t)=1 \mid X(s)=1]$$
$$= \mathbb{P}[N(t) \text{ is even}] = \sum_{k=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^{2k}}{(2k)!} = e^{-\lambda t}\left(\frac{e^{\lambda t}+e^{-\lambda t}}{2}\right)$$

Solving for $P_t(1,-1)$, $P_t(-1,1)$ and $P_t(-1,-1)$, we get that
$$P_t = \frac{1}{2}\begin{array}{c c} \begin{array}{cc} -1 & 1 \end{array} & \\ \begin{pmatrix} 1+e^{-2\lambda t} & 1-e^{-2\lambda t} \\ 1-e^{-2\lambda t} & 1+e^{-2\lambda t} \end{pmatrix} & \begin{array}{c} -1 \\ 1 \end{array} \end{array}$$

- Alternately, we can write for $t, \delta > 0$, and $\Pi_t = \begin{pmatrix} \Pi_t(1) \\ \Pi_t(-1) \end{pmatrix}$

$$\Pi_{t+\delta}(-1) = \Pi_t(-1)(1-\lambda\delta) + \Pi_t(1)\,\lambda\delta + O(\delta^2)$$
$$\Pi_{t+\delta}(1) = \Pi_t(-1)(\lambda\delta) + \Pi_t(1)(1-\lambda\delta) + O(\delta^2)$$

As before we can compute $(\Pi_{t+\delta}(x) - \Pi_t(x))/\delta$ and take $\lim \delta \searrow 0$ to get $\frac{d\Pi_t}{dt} = \Pi_t^T Q$, where $Q = \begin{pmatrix} -\lambda & \lambda \\ \lambda & -\lambda \end{pmatrix}$

Solving this, we get $\Pi_t^T = \Pi_0^T e^{Qt}$, where one can check via computing the e-values of Q that $e^{Qt} = \frac{1}{2}\begin{pmatrix} 1+e^{-2\lambda t} & 1-e^{-2\lambda t} \\ 1-e^{-2\lambda t} & 1+e^{-2\lambda t} \end{pmatrix}$

- Thus in both cases, we managed to derive P_t by writing $\frac{d\Pi_t^T}{dt} = \Pi_0^T Q$ and solving the system to get $\Pi_t^T = \Pi_0^T P(t)$. Before formalizing this, we see another example, which generalizes the above 2.

Eg (Uniform CTMC) Let $\{Y_n\}_{n\in\mathbb{N}}$ be a DTMC on countable state-space $\mathcal{X}$, with transition matrix K, and let $\{T_n\}_{n\in\mathbb{N}}$ be the arrival times of a PP(λ) process $N(t)$ of rate λ. Then the process $X(t) = Y_{N(t)}$ is called a uniform MC with Poisson clock $N(t)$ and subordinate chain Y_n

- Thus $X(T_n) = Y_n \;\forall\, n\in\mathbb{N}$, and $X(t) = X(t^-)$ if $t \notin \{T_n\}_{n\in\mathbb{N}}$. Note also that T_n is not necessarily a discontinuity pt of $X(t)$, since Y_n can equal Y_{n-1}.

- Now we have $P_t = \sum_{n=0}^{\infty} \left(\frac{e^{-\lambda t}(\lambda t)^n}{n!} \right) \cdot K^n$

- Note also that $\forall\, t, \delta > 0$, we have $\forall\, y \in \mathcal{X}$

$$\Pi_{t+\delta}(y) = \lambda\delta \left(\sum_{x\in\mathcal{X}} \Pi_t(x) K(x,y) \right) + (1-\lambda\delta)\Pi_t(y) + O(\delta^2)$$

$$\Rightarrow \frac{d\Pi_t}{dt} = \lim_{\delta\downarrow 0} \frac{\Pi_{t+\delta} - \Pi_t}{\delta} = \lambda(K-I)\Pi_t$$

Solving we get $\Pi_t = \Pi_0 e^{\lambda t(K-I)}$, where we have $e^{\lambda t(K-I)} = \sum_{n=0}^{\infty} e^{-\lambda t} \frac{(\lambda t K)^n}{n!} = P_t$

Now we try and formalize these ideas . . .

Defn (Stochastic Semigroup) $\{P_t\}_{t\geq 0}$ is said to be a stochastic semigroup on $\mathcal{X}$ if $\forall s,t \geq 0$

(i) P_t is stochastic matrix, i.e., $\sum_{y\in\mathcal{X}} P_t(x,y) = 1 \ \forall x \in \mathcal{X}$

(ii) $P_0 = I$, (iii) $P_{t+s} = P_t P_s \ \forall s,t \geq 0$

- A stochastic semigroup P is called standard if it is continuous at the origin, i.e., $\lim_{\delta\to 0} P_\delta = P_0 = I$ (pointwise convergence ?)

Then we have the following 2 properties:

i) P_t is continuous, i.e., $\lim_{\delta\to 0} P_{t+\delta} = P_t \quad \forall t \geq 0$

ii) $\forall x \in \mathcal{X}$, $\exists$ $-Q(x,x) = Q(x) \triangleq \lim_{\delta\to 0} \frac{1-P_\delta(x,x)}{\delta}$, $Q(x,y) \triangleq \lim_{\delta\to 0} \frac{P_\delta(x,y)}{\delta}$

The proof is purely analytical (see Brémaud, Ch8, Thm 2.1, and not crucial for our purposes, so we take it as a fact.

Defn (Infinitesimal Generator) - For a CTMC $\{X(t)\}$ on $\mathcal{X}$ with stochastic semigroup P_t, its infinitesimal generator is given by $Q = \lim_{\delta\downarrow 0} \frac{P_\delta - I}{\delta}$ $(\because P_0 = I)$

- The generator Q is thus the derivative of $P_t - I$ at $t=0$, and can be found from P_t. On the other hand, P_t can usually also be found from Q (as $P_t = e^{tQ}$) in most cases...

Eg (Birth-Death Process) A continuous-time birth-death process $X(t)$ is a CTMC taking values in $\mathbb{N}$ s.t. $\forall\, t, \delta > 0$ and $i \in \mathbb{N}$

$\mathbb{P}[X(t+\delta) = i+1 \mid (t) = i] = \lambda_i \delta + O(\delta^2)$, $\mathbb{P}[X(t+\delta) = i-1 \mid X(t) = i] = \mu_i \delta + O(\delta^2)$

$\mathbb{P}[X(t+\delta) = i \mid X(t) = i] = 1 - (\lambda_i + \mu_i)\delta + O(\delta^2)$

and all other transitions have probability $O(\delta^2)$

Given $\{\lambda_i, \mu_i\}_{i \geq 1}$ and λ_0, intuitively we would say $X(t)$ has a generator $Q(i, i+1) = \lambda_i$, $Q(i, i-1) = \mu_i \mathbb{1}_{\{i \geq 1\}}$, $Q(i,j) = 0 \;\forall j \notin \{i-1, i, i+1\}$

However, if $\lambda_i, \mu_i \uparrow \infty$ as $i \uparrow \infty$, such a limit may not exist. We need to be somewhat careful dealing with such cases.

Defn - Consider semigroup P_t with generator $Q = \lim_{\delta \downarrow 0} \frac{P_\delta - I}{\delta}$

- P_t is stable iff $(-Q(x,x) =)\; Q(x) = \lim_{\delta \downarrow 0} \frac{1 - P_\delta(x,x)}{\delta} < \infty \;\; \forall x \in \mathcal{X}$

- P_t is conservative iff $(-Q(x,x) =)\; Q(x) = \sum_{y \neq x} Q(x,y) \;\; \forall x \in \mathcal{X}$

- Note that for any δ, by defn of the stoch semigroup

$$\forall x, \sum_{y \in \mathcal{X}} P_\delta(x,y) = 1 \Rightarrow \frac{1 - P_\delta(x,x)}{\delta} = \frac{\sum_{y \neq x} P_\delta(x,y)}{\delta}$$

Thus if $\lim_{\delta \downarrow 0} \sum_{y \neq x} P_\delta(x,y) = \sum_{y \neq x} \lim_{\delta \downarrow 0} P_\delta(x,y)$, then P is stable and conservative. We assume henceforth that Q is stable and conservative – note though that checking this for a CTMC (for example, a birth-death chain) is non-trivial

Kolmogorov's Differential Equations

- Given a standard stochastic semigroup P_t, we can write $\forall t, s$

$$\frac{P_{t+\delta}-P_t}{\delta} = P_t \frac{P_\delta - I}{\delta} = \frac{P_\delta - I}{\delta} P_t$$

Assuming the limit $\delta \downarrow 0$ exists, we get two systems of diff eqs

i) $\dfrac{dP_t}{dt} = P_t Q$ (Forward diff system) ii) $\dfrac{dP_t}{dt} = Q P_t$ (Backward diff system)

In more detail, $\forall x, y \in \mathcal{X}$, we have the diff eqns

Forward Eqns i) $$\frac{d P_t(x,y)}{dt} = \sum_{z\in\mathcal{X}} P_t(x,z) Q(z,y) = -P_t(x,y) Q(y) + \sum_{z\in\mathcal{X}} P_t(x,z) Q(z,y)$$

Backward Eqns ii) $$\frac{dP_t(x,y)}{dt} = \sum_{z\in\mathcal{X}} Q(x,z) P_t(z,y) = -Q(x) P_t(x,y) + \sum_{z\in\mathcal{X}} Q(x,z) P_t(z,y)$$

- If $\mathcal{X}$ is finite, then subject to $P(0) = I$, the above systems have a unique soln $P(t) = e^{tQ} = \sum_{k=0}^{\infty} \frac{t^k Q^k}{k!}$

— For verifying this, the main thing that one must check is that e^{tQ} is defined. For this we have the following -

- Lemma - For any $n \times n$ matrix A with $A(i,j) \in \mathbb{R}$, and for all $t > 0$, the series $\sum_{k=0}^{\infty} t^k A^k / k!$ converges componentwise (ie, for all $i, j \in [n]^2$)

Pf - let $A_k(i,j) = (A^k)_{i,j}$, and define $\Delta = \max_{i,j} |A_1(i,j)|$. Check via induction that $|A_k(i,j)| \leq \Delta^k n^{k-1}$. Hence $\forall i, j \in [n]^2$, we have $A_k(i,j) t^k / k! \leq \frac{1}{n} (n\Delta t)^k / k! \Rightarrow (e^{At})_{ij} \leq e^{n\Delta t}/n$ ∎

- What about when $\mathcal{X}$ is countable? This gets more technical, so we state the main results without proof

<u>Thm</u> - Let P_t be a standard stochastic semigroup.

i) If P_t is <u>stable and conservative</u>, then $\frac{dP_t}{dt} = QP_t$ (i.e., we can take limits to get Kolmogorov's backward system)

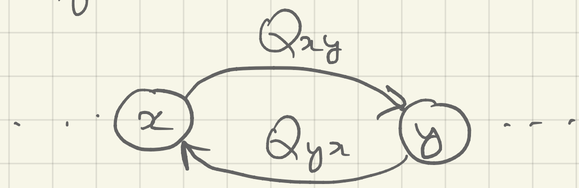
ii) If in addition $\sum_{k \in \mathcal{X}} P_t(x,k) Q(k) < \infty \; \forall x \in \mathcal{X}$, then also $\frac{dP_t}{dt} = P_t Q$ (i.e., Kolmogorov's forward system is satisfied)

iii) Finally let π_t denote the distribution of $X(t)$ at any $t \geq 0$. Assuming the above conditions, and also, that $\forall t \geq 0$ we have $\sum_{x \in \mathcal{X}} Q(x) \pi_t(x) < \infty$. Then we have

$$\frac{d\pi_t^T}{dt} = \pi_t^T Q, \text{ i.e. } \forall x \in \mathcal{X}, \; \frac{d\pi_t(x)}{dt} = -\pi_t(x) Q(x) + \sum_{y \neq x} \pi_t(y) Q(y,x)$$

- To summarize, assuming $Q(x) < \infty$ and $Q(x) = \sum_{y \neq x} Q(x,y)$ (i.e., Q is stable and conservative), we can solve the backward eqns to obtain $P_t = e^{Qt}$. Thus Q is a sense completely defines the CTMC.

- $Q(x,y)$ is sometimes referred to as the transition rate from x to y (for $x \neq y$), as it represents the rate of 'probability' flowing from x to y (i.e., $P_\delta(x,y) = Q(x,y)\delta + O(\delta^2)$). This can be represented by a transition rate diagram ≡



Defn (Irreducibility) - A CTMC $X(t)$ with generator Q is irreducible if $P_t(x,y) > 0$ for any $t > 0$, and all $x,y \in \mathcal{X}$

- In fact, for any $x,y \in \mathcal{X}$, $P_t(x,y) > 0 \ \forall t$, or $P_t(x,y) = 0 \ \forall t$

• **Defn** (Stationary Distribution) A stochastic vector π (ie, with $\pi(x) \geq 0 \ \forall x \in \mathcal{X}$ and $\sum_{x\in\mathcal{X}} \pi(x) = 1$) is a stationary distr of a CTMC (Q, P_t) if $\pi^T P_t = \pi^T \ \forall \ t \geq 0$.
Moreover if Q is stable and conservative, then π satisfies the global balance eqn $\pi^T Q = 0$

Given the above defn, we can state a convergence theorem for CTMCs

Thm (CTMC Convergence Theorem) For an irreducible CTMC (P_t, Q)

i) If stationary dist π exists, then it is unique, and moreover

$$P_t(x,y) \xrightarrow{t\to\infty} \pi(y) \quad \forall \ x,y \in \mathcal{X}$$

2) If no stationary π exists, then $P_t(x,y) \xrightarrow{t\to\infty} 0 \ \forall x,y \in \mathcal{X}$

Pf Sketch (G&S, Ch6, Thm 21) For any $k > 0$, define skeleton DTMC $Y_n = X(nk)$.
Note Y_n is irreducible, positive recurrent ($\because X(t)$ irreducible and $P_t(x,y) \xrightarrow{t\to\infty} \pi(y)$) and aperiodic ($\because P_t(x,x) > 0$) $\Rightarrow Y_n$ has unique stationary dist $\pi_{(k)}$, and $P_{nk}(x,y) \to \pi_{(k)}(y)$
Now consider $k_1, k_2 \in \mathbb{Q}$: since $k_1 n = k_2 n'$ infinitely often $\Rightarrow \pi_{(k_1)} = \pi_{(k_2)}$. For any other $t \notin \mathbb{Q}$, we can complete the proof via continuity arguments. $\blacksquare$

Thm (CTMC Ergodic Theorem) - For irreducible CTMC $X(t)$ with stationary dist^n π, we have $\lim_{T\to\infty} \frac{1}{T} \int_0^T f(X(s))ds = \sum_{x\in\mathcal{X}} f(x)\pi(x)$ a.s. $\forall f$ s.t $\mathbb{E}_\pi[|f(X)|] < \infty$

Pf Sketch - Similar to DTMC (via renewal cycles)

CTMCs via Embedded Chains

- An alternate approach to constructing CTMCs is by constructing them from DTMCs. There are two ways to do this:

i) The Jump Chain - This exists for any CTMC

ii) The Uniformized Chain - This exists when $\sup_{x \in \mathcal{X}} Q(x) < \infty$

The Jump Chain

- We construct a process $X(t)$ on some countable $\mathcal{X}$, for $t \in \mathbb{R}_+$ as follows -
  - Start with a DTMC $\{Y_n\}_{n \in \mathbb{N}}$ on $\mathcal{X}$, with $Y_0 \sim \pi_0$, and transition prob matrix $A = \{A(x,y)\}$. We assume that $A(x,x) = 0 \ \forall x \in \mathcal{X}$; in other words, Y_n has no self-loops, but always 'jumps' to a new state.
  - Next suppose we are given a sequence $\{E_n\}_{n \in \mathbb{N}}$ of iid $\text{Exp}(1)$ r.v.s (ind of Y_n), and a function $\{\gamma(x);\ x \in \mathcal{X}\}$ of inverse holding times for each state. Essentially, whenever $X(t)$ reaches a state $x \in \mathcal{X}$, we want it to stay there for time $W \sim \text{Exp}(\gamma(x))$ before jumping to some $y \neq x$.

- We now construct the chain as follows.
  - Let $X(0) = Y_0 \sim \Pi_0$ and $T_0 = 0$.
  - Define $W_0 = E_0/\gamma(Y_0) \sim \mathcal{E}xp(\gamma(Y_0))$
  - Set $T_1 = T_0 + W_0$, and $X(T_1) = Y_1$
  - Subsequently for any $k \geq 1$, we define

$$W_k = E_k/\gamma(Y_k), \; T_{k+1} = T_k + W_k, \; X(T_{k+1}) = Y_{k+1}$$

- Define $T_\infty = \lim_{k\to\infty} T_k$. Then we can write

$$X(t) = \sum_{k=0}^{\infty} Y_k \mathbb{1}\{t \in [T_k, T_{k+1})\} \quad \forall t \in [0, T_\infty)$$

- It is not hard to check that the above process is indeed Markovian. Moreover, we could also allow $\lambda(x) = 0$ to model absorbing states, or $\lambda(x) = \infty$ to model states visited instantaneously. For the following, however we restrict to $\lambda(x) \in (0, \infty)$ $\forall x \in \mathcal{X}$.

- One potential problem still is that T_∞ could be finite (and hence $X(t)$ is only defined on $t \in [0, T_\infty] \subset \mathbb{R}$)

- <u>Defn</u> - The process $X(t)$ is said to be explosive if $\mathbb{P}_x[T_\infty < \infty] > 0$ for some $X(0) = x$, and regular if $\mathbb{P}_x[T_\infty < \infty] = 0$ for all $X(0) = x \in \mathcal{X}$.

- As an example, consider a birth process with $\gamma(x) = x^2$

Thm - For any $x \in \mathcal{X}$, given $\{Y_n\}_{n\in\mathbb{N}}$ and $\{\gamma(x)\}$ as above

$$\mathbb{P}_x[T_\infty < \infty] = \mathbb{P}_x\left[\sum_n \frac{1}{\gamma(Y_n)} < \infty\right]$$

In other words, $X(t)$ is regular iff $\sum_n \gamma(Y_n)^{-1} = \infty$ a.s.

Moreover, this holds whenever one of the following hold
i) $\mathcal{X}$ is finite, ii) $\gamma(x) \leq \gamma < \infty \ \forall x \in \mathcal{X}$,
iii) Given $A \subset \mathcal{X}$ the transient states of Y_n, we have $\forall x \in \mathcal{X}$, $\mathbb{P}_x[Y_n \in A \ \forall n \in \mathbb{N}] = 0$

We first need a property of Exponential r.v

Proposition - If $\{E_n\}$ are independent Exponential r.v s.t $E_i \sim \mathrm{Exp}(\lambda_i) \ \forall i \in \mathbb{N}$. Then

$$\sum_{n\in\mathbb{N}} E_n < \infty \text{ a.s. iff } \sum_{n\in\mathbb{N}} \lambda_i^{-1} < \infty$$

Pf of theorem - By construction, we have $T_\infty = \sum_{n\in\mathbb{N}} E_n/\gamma(Y_n)$

This is a sum of indep Exponential r.v.s, and by the above prop^n, $\mathbb{P}[T_\infty < \infty \mid \{Y_n\}] = \begin{cases} 1 & \text{if } \sum \gamma(Y_n)^{-1} < \infty \\ 0 & \text{if } \sum \gamma(Y_n)^{-1} = \infty \end{cases}$

Thus $\mathbb{P}[T_\infty < \infty] = \mathbb{P}_x\left[\sum_n \lambda(Y_n)^{-1} < \infty\right]$

Now we want to verify the sufficient conditions

- For (i), note that $\mathcal{X}$ finite means $\gamma(x) \leq \gamma < \infty$ $\forall x \in \mathcal{X}$. Thus its enough to verify (ii)
- For (ii), we have $\sum_n \gamma(Y_n)^{-1} \geq \sum_n \gamma^{-1} = \infty$
- For (iii), suppose $\mathbb{P}[Y_n \in A \ \forall n] = 0$ implies that $\exists$ some $x_0 \in \mathcal{X} | A$ s.t. x is hit infinitely often. Suppose $Y_{n_j} = x$ for some set n_j, $j \in \{1, 2, \dots\}$. Then

$$\sum_{n \in \mathbb{N}} \gamma(Y_n)^{-1} \geq \sum_{j \in \mathbb{N}_+} \gamma(Y_{n_j})^{-1} = \sum_{j \in \mathbb{N}_+} \gamma(x_0) = \infty$$

$\blacksquare$

- Assume now we are given DTMC transition matrix A and holding times $\{\gamma(x)\}_{x \in \mathcal{X}}$ which are non-explosive

<u>Proposition</u> $\forall x, y \in \mathcal{X}$, $t \geq 0$, we have

$$P_t(x,y) = e^{-\gamma(x)t} \mathbb{1}_{\{x=y\}} + \int_0^t \gamma(x) e^{-\gamma(x)s} \left(\sum_{z \neq x} A(x,z) P_{t-s}(z,y) \right) ds$$

<u>Pf</u> - $P_t(x,y) = \mathbb{P}[X(t) = y \mid X(0) = x]$

$$= \mathbb{P}[X(t)=y, W_0 > t \mid X(0) = x] + \mathbb{P}[X(t)=y, W_0 \leq t \mid X(0)=x]$$

Moreover by construction, we have

$$\mathbb{P}[X(t)=y, W_0 > t \mid X(0) = x] = e^{-\gamma(x)t} \mathbb{1}_{\{y=x\}}$$

and

$$\mathbb{P}[X(t)=y, W_0 \leq t \mid X(0) = x] = \sum_{z \neq x} \mathbb{P}[X(t)=y, Y_1 = z, W_0 \leq t \mid X(0)=x]$$

$$= \sum_{z \neq x} \int_0^t e^{-\gamma(x)s} \gamma(x) . A(x,z) . P_{t-s}(z,y) ds$$

• Now given the expression for $P_t(x,y)$, we get the following

i) $\lim_{t\downarrow 0} P_t(x,y) = \mathbb{1}_{\{x=y\}} \Rightarrow \lim_{t\downarrow 0} P_t = I \Rightarrow P_t$ is standard

ii) $$P_t(x,y) = e^{-t\gamma(x)}\left[\mathbb{1}_{\{x=y\}} + \int_0^t \gamma(x) e^{\gamma(x)u} \sum_{z\neq x} A(x,z) P_u(z,y)\,du\right]$$
(substituting $u = t-s$)

$$\Rightarrow \frac{dP_t(x,y)}{dt} = -\gamma(x) P_t(x,y) + e^{-\gamma(x)t}\gamma(x)e^{\gamma(x)t}\left(\sum_{z\neq x} A(x,z)P_t(z,y)\right)$$
$$= \gamma(x)\left(-P_t(x,y) + \sum_{z\neq x} A(x,z)P_t(z,y)\right)$$

$$\Rightarrow \frac{dP_t}{dt} = QP_t, \text{ where } Q(x,z) = \begin{cases} -\gamma(x) & ; z=x \\ \gamma(x)A(x,z) & ; z\neq x \end{cases}$$
(Kolmogorov's Backward Eqn)

$$= D_\gamma (A-I)$$

$$\operatorname{diag}(\gamma) = \begin{pmatrix} \gamma(1) & 0 & \cdots & 0 \\ & \gamma(2) & & 0 \\ 0 & & \ddots & \\ & & & \gamma(n) \end{pmatrix}$$

Moreover $Q(x,x) = -\gamma(x) < \infty\ \forall x$, and $\sum_{z\neq x}\gamma(x)A(x,z) = \gamma(x) = -Q(x,x)$

Thus, Q is stable and conservative.

iii) The formula for $P_t(x,y)$ is based on conditioning on the first jump (and hence yields the backward DE). To get the forward DE, we need to condition on the last jump – for this to exist, we need the system to not be explosive. Now conditioning on the last jump, we get
$$P_t(x,y) = e^{-\gamma(x)t}\mathbb{1}_{\{x=y\}} + \int_0^t\left(\sum_{z\neq y} e^{-\gamma(z)s} P_s(x,z)Q(z,y)\right)ds$$
Differentiate to derive the Kolmogorov forward equation

iv) Given a (stable + conservative) infinitesimal generator Q, we can derive the jump chain parameters (A, γ) as

$$\gamma(x) = -Q(x,x) \;\forall x \in \mathcal{X}, \quad A(x,y) = Q(x,y)/\gamma(x) \;\; \forall x,y \in \mathcal{X}$$

Similarly given a sample path $X(t)$ of a CTMC, we can obtain the subordinate jump DTMC by tracking the sequence of unique states of $X(t)$ (so $Y_0 = X(0)$, $T_1 = \inf\{t>0 \mid X(t) \neq X(0)\}$, $Y_1 = X_{T_1}$, $T_2 = \inf\{t > T_1 \mid X(t) \neq X(T_1)\}$, $Y_2 = X_{T_2}, \ldots$). Moreover, the holding times $W_i = T_i - T_{i-1}$ can be used to give the underlying driving clock process as $E_i = W_i \cdot \lambda(X_{T_i})$

The Uniformized Chain

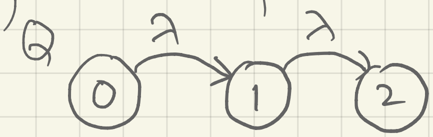
- Earlier we saw a uniform $X(t) = Y_{N(t)}$, where Y_n is a DTMC with transition matrix K (where $K(x,x)$ can be ≥ 0, ie, self loops are allowed), and $N(t) \sim PP(\lambda)$. The associated stochastic semigroup is $P_t(x,y) = \sum_{n=0}^{\infty} \frac{e^{-\lambda t}(\lambda t)^n}{n!} K^n(x,y)$
- We can differentiate to check that
$$\frac{dP_t(x,y)}{dt} = -\lambda e^{-\lambda t} \mathbb{1}_{\{x=y\}} + \sum_{n=1}^{\infty} (-\lambda^2 t + n\lambda) \frac{(\lambda t)^{n-1}}{n!} e^{-\lambda t} K^n(x,y)$$
Setting $t=0$, we get $dP_0(x,y)/dt = \lambda(K(x,y) - \mathbb{1}_{\{x=y\}}) \Rightarrow Q = \lambda(K - I)$
- To determine the jump chain (A, γ) associated with the Uniform CTMC (K, λ), we have $\gamma(x) = -Q(x,x) = \lambda(1-K(x,x))$, $A(x,x) = 0$ and $A(x,y) = \frac{Q(x,y)}{\gamma(x)} = \frac{K(x,y)}{1-K(x,x)}$. Note that $A(x,y) = \mathbb{P}[Y_{n+1} = y \mid Y_n = x, Y_{n+1} \neq x]$

- Note that $-Q(x,x) = \lambda(1-K(x,x)) \le \lambda \Rightarrow \sup_{x\in\mathcal{X}}(-Q(x,x)) \le \lambda < \infty$. Moreover, given any Q s.t. $\sup_{x\in\mathcal{X}}(-Q(x,x)) < \infty$, we can obtain a uniform MC with $\lambda = \sup_{x\in\mathcal{X}}(-Q(x,x))$, $K(x,x) = 1 + Q(x,x)/\lambda$, $K(x,y) = Q(x,y)/\lambda$

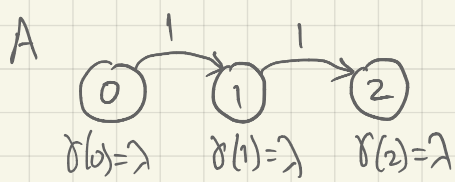
Thus **CTMC Q is uniformizable iff $\sup_{x\in\mathcal{X}}(-Q(x,x)) < \infty$.**

Examples

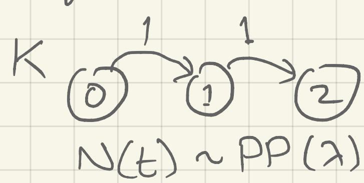
i) Poisson process $PP(\lambda) \equiv Q(x,y) = \lambda \mathbb{1}_{\{y=x+1\}} - \lambda \mathbb{1}_{\{y=x\}}$



rate diagram

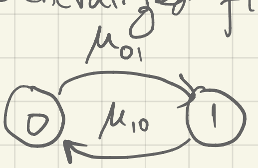


jump chain

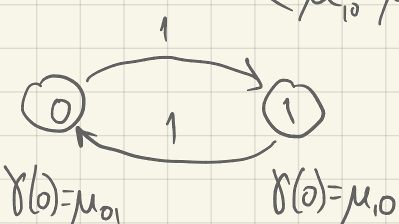


uniformized chain

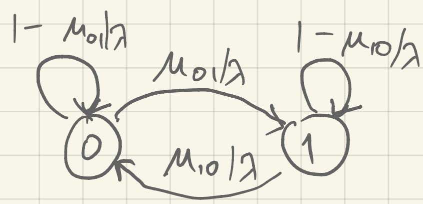
ii) Generalized flip-flop - $Q = \begin{pmatrix} -\mu_{01} & \mu_{01} \\ \mu_{10} & -\mu_{10} \end{pmatrix}$



rate diagram



jump chain

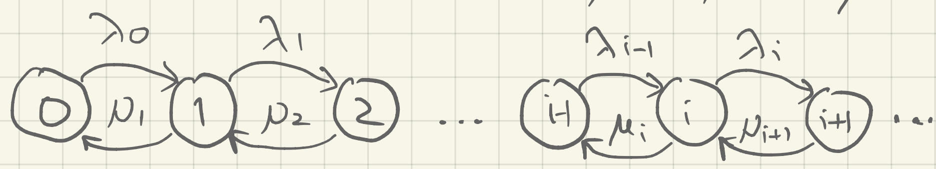


$\lambda = \max(\mu_{01}, \mu_{10})$

uniformized chain

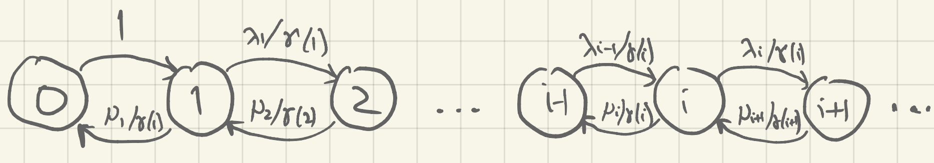
iii) Generalized birth-death process - $Q(i,i+1) = \lambda_i$, $Q(i,i-1) = \mu_i$

rate diagram



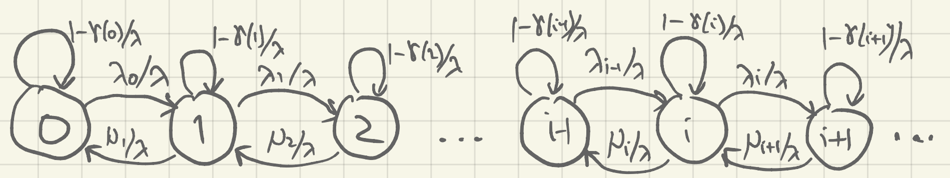
jump chain

$\gamma(i) = \mu_i + \lambda_i \ \forall i$



uniformized chain

$\lambda = \sup_i(\lambda_i + \mu_i) \underbrace{< \infty}_{\text{assume}}$



Finally we want to understand stationary distributions.

- Recall: Given CTMC (P_t, Q), π is a stationary distr iff
  i) $\pi^T P_t = \pi^T \quad \forall t \geq 0$
  ii) (Assuming Q is stable, conservative) $\pi^T Q = 0$

The CTMC is ergodic if it is irreducible, and $\pi(x) > 0$ for some $x \in \mathcal{X}$

We now want to relate this to the jump chain (A, γ) and uniform chain (K, λ).

Recall $Q^j = D_\gamma(A - I)$ for the jump chain, $Q^u = \lambda(K - I)$ for the uniformized chain. Thus the stationary distr satisfies

- Uniformized Chain -

$$\pi^T \lambda (K - I) = 0 \Rightarrow \pi^T K = \pi^T$$

In other words, stationary dist of CTMC $Q^u \equiv$ stationary dist of DTMC K

Moreover CTMC $X(t)$ with generator Q is ergodic $\Longleftarrow$ $X(t)$ is uniformizable with uniform chain (λ, K), and K is ergodic

- Jump Chain -

$$\pi^T D_\gamma (A - I) = 0 \Rightarrow \pi(x)\gamma(x) = \sum_{y \neq x} \pi(y)\gamma(y) A(y,x) \quad \forall x$$

Moreover, the chain is ergodic if $\exists$ soln π st $\sum_{x \in \mathcal{X}} \pi(x) < \infty$

Stationary Distr - Calculation & Examples

To summarize the formulations of CTMCs -

Generator	Jump Chain	Uniformized Chain
• $Q(x,y) = \lim_{s\to 0} P_s(x,y)/s$	• Holding rates - $\gamma(x)$, $x \in \mathcal{X}$	• (Uniform) rate - λ
$Q(x,x) = -\sum_{y\neq x} Q(x,y) < \infty$	Subordinate DTMC - A	Subordinate DTMC - K
(stable + conservative)	- $A(x,x) = 0 \ \forall\, x \in \mathcal{X}$	• $Q = \lambda(K - I)$
• $\frac{dP_t}{dt} = QP_t = P_tQ$	• $D_\gamma = \text{diag}(\gamma(x))$	• Only valid for Q s.t.
• $\pi_t^T = \pi_0^T P_t$, $\frac{d\pi_t^T}{dt} = \pi_t^T Q$	$Q = D_\gamma(A - I)$	$\sup_x \sum_{y\neq x} Q(x,y) < \infty$
• $\pi^T Q = 0 \Leftrightarrow \pi^T P_t = \pi^T$	• Non-explosive $\equiv \mathbb{P}[T_\infty < \infty] < 1$	• $\pi(x) = \sum_y \pi(y) K(y,x)$
	i) $\mathcal{X}$ finite, ii) $\sup \lvert Q(x,x)\rvert < \infty$	(i.e., $\pi^T K = \pi^T$)
	iii) A is ergodic	
	• $\pi(x)\gamma(x) = \sum_{y\neq x} \pi(y)\gamma(y)A(y,x)$	

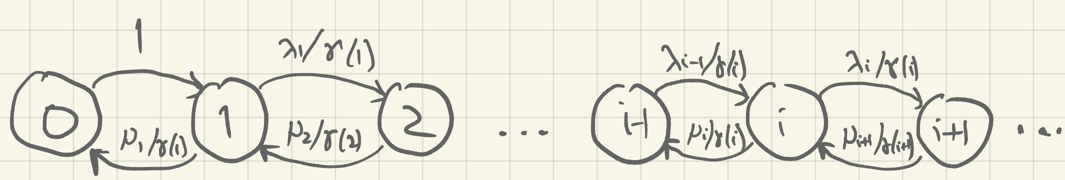
Eg - (General Birth-Death Chain) $\mathcal{X} = \mathbb{N}_0$, in state i new arrival after time $Exp(\lambda_i)$, departure after time $Exp(\mu_i)$

rate diagram

$$Q(x,y) = \begin{cases} \lambda_x ; & y = x+1 \\ \mu_x ; & y = x-1, x > 0 \\ 0 ; & \text{ow} \end{cases}$$

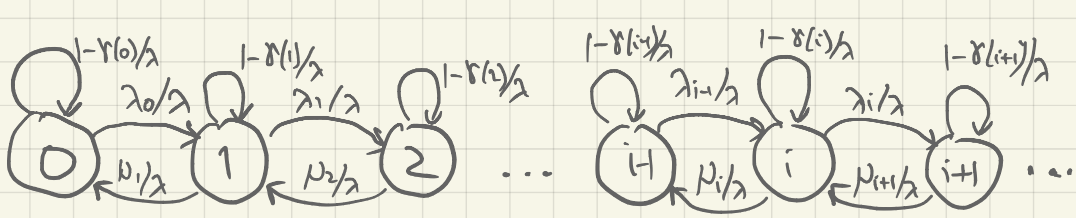
jump chain

$\gamma(i) = \mu_i + \lambda_i \ \forall\, i$



uniformized chain

$\lambda = \sup_i (\lambda_i + \mu_i) < \infty$



- Assuming this has a stationary dist^n Π, we can solve for it in 3 ways

i) Using the rate matrix - Π stationary if $\sum_{x=0}^{\infty} \Pi(x) = 1$ and $\Pi^T Q = 0$

$$\Rightarrow (\Pi(0)\ \Pi(1)\ \dots) \begin{pmatrix} -\lambda_0 & \lambda_0 & 0 & \dots \\ \mu_1 & -(\lambda_1+\mu_1) & \lambda_1 & 0 \dots \\ 0 & \mu_2 & -(\lambda_2+\mu_2) & \lambda_2 & 0 \dots \\ \vdots & & & & \ddots \end{pmatrix} = 0 \Rightarrow \Pi(0)\lambda_0 = \Pi(1)\lambda_1, \text{ and } \forall i \geq 1$$

$$(\lambda_i + \mu_i)\Pi(i) = \lambda_{i-1}\Pi(i-1) + \mu_{i+1}\Pi(i+1)$$

Thus $\Pi(2)\mu_2 = (\lambda_1+\mu_1)\Pi(1) - \lambda_0\Pi(0) = \lambda_1\Pi(1)$, $\mu_3\Pi(3) = (\lambda_2+\mu_2)\Pi(2) - \lambda_1\Pi(1) = \lambda_2\Pi(2)$

Now by induction, we can show $\Pi(i)\mu_i = \Pi(i-1)\lambda_i$ $\forall i \geq 1$

Let $\alpha_0 = 1$, $\alpha_x = \lambda_{x-1}/\mu_x$ $\Rightarrow$ $\Pi(x) = \Pi(0) . \underbrace{(\alpha_0 \alpha_1 \dots \alpha_x)}^{\triangleq \beta_x}$ $\forall x \in \mathbb{N}_0$

Thus $\Pi(x) = \dfrac{\alpha_0 \alpha_1 \dots \alpha_x}{Z} \triangleq \dfrac{\beta_x}{Z}$, where partition fn (ie, normalization) $Z = \sum_{x=0}^{\infty} \beta_x$

ii) Using the jump chain - $\Pi(x)\gamma(x) = \sum_{y \neq x} \Pi(y)\gamma(y)A(y,x)$ $\forall x \geq \mathbb{N}_0$

$\Rightarrow$ $\Pi(0)\lambda_0 = \mu_1\Pi(1)$, $(\lambda_i+\mu_i)\Pi(i) = \lambda_{i-1}\Pi(i-1) + \mu_{i+1}\Pi(i+1)$ $\forall i \geq 1$

Now we solve as before to get $\Pi(x) = \beta_x/Z$, $Z = \sum_{x=0}^{\infty} \beta_x = \sum_{x=0}^{\infty}\left(\prod_{i=0}^{x} \alpha_i\right)$

iii) Using the uniform chain - $\lambda \geq \max_{i \geq 0}(\mu_i + \lambda_i)$ (assume $< \infty$)

Moreover $\Pi(x)\,\mu_x/\lambda = \Pi(x-1)\lambda_{x-1}/\lambda \Rightarrow \Pi(x) = \alpha_x \Pi(x-1) = \left(\prod_{i=0}^{x}\alpha_i\right)\Pi(0)$

(This follows from the balance eqns for DTMC K - Note also K is reversible)

Again we get $\Pi(x) = \dfrac{\beta_x}{Z} = \dfrac{\prod_{i=0}^{x}\alpha_i}{Z}$, $Z = \sum_{x=0}^{\infty}\left(\prod_{i=0}^{x}\alpha_i\right)$

- Thus in all cases we get $\Pi(x) = \beta_x/Z$, $\beta_x = \prod_{i=0}^{x-1}\left(\dfrac{\lambda_i}{\mu_{i+1}}\right)$, $Z = \sum_{x=0}^{\infty}\beta_x$

For this to be defined, we need $Z < \infty \Rightarrow$ $\boxed{1 + \dfrac{\lambda_0}{\mu_1} + \dfrac{\lambda_0\lambda_1}{\mu_1\mu_2} + \dfrac{\lambda_0\lambda_1\lambda_2}{\mu_1\mu_2\mu_3} + \dots < \infty}$ $\circledast$

- Finally to guarantee that π exists, we need $\circledast$ + non-explosiveness of the MC. We can ensure this in two ways

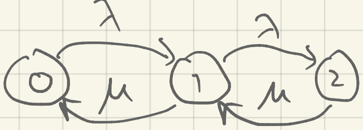
i) $\sup_i (\lambda_i + \mu_i) < \infty$ (ie, uniformizable chain), or

ii) The jump chain DTMC A is positive recurrent : for this, we can check balance eqns as above to get the condn

$$1 + \frac{1 \cdot \gamma(1)}{\mu_1} + \frac{\cancel{\gamma(1)}}{\mu_1} \cdot \frac{\lambda_1}{\cancel{\gamma(1)}} \cdot \frac{\gamma(2)}{\mu_2} + \frac{\lambda_1 \lambda_2 \gamma(3)}{\mu_1 \mu_2 \mu_3} + \dots < \infty \Rightarrow \left(1 + \frac{\lambda_1}{\mu_1} + \frac{\lambda_1 \lambda_2}{\mu_1 \mu_2} + \frac{\lambda_1 \lambda_2 \lambda_3}{\mu_1 \mu_2 \mu_3} + \dots\right) < \infty$$

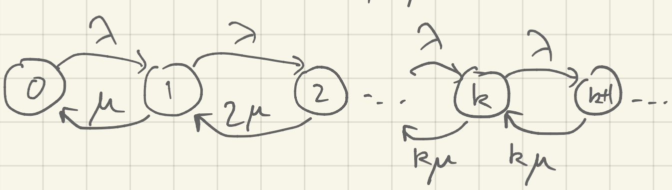
This has many special cases

i) $\lambda_i = \lambda$, $\mu_i = \mu \mathbb{1}_{\{i>0\}}$ $\forall i$ M/M/1 queue



let $\rho = \frac{\lambda}{\mu} \Rightarrow Z = \sum_{i=0}^{\infty} \rho^i < \infty$ iff $\rho < 1$, $\pi(i) = (1-\rho)\rho^i$

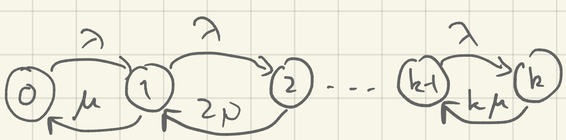
ii) $\lambda_i = \lambda$, $\mu_i = (i \wedge k)\mu$ $\forall i$ $(\rho = \frac{\lambda}{\mu})$ M/M/k queue



$$Z = \sum_{i=0}^{k-1} \frac{\rho^i}{i!} + \frac{\rho^k}{k!} \sum_{i=0}^{\infty} \left(\frac{\rho}{k}\right)^i < \infty \text{ iff } \rho < k \text{ (ie, } \lambda < k\mu)$$

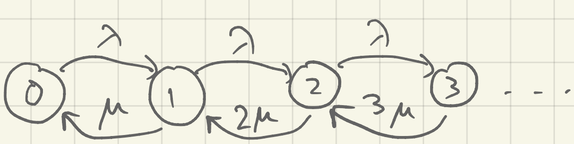
$$\pi(x) = \frac{\rho^x}{Z\, x!} \text{ if } x \le k, \quad \pi(x) = \frac{\rho^k}{Z\, k!} \cdot \left(\frac{\rho}{k}\right)^{x-k}$$

iii) $\lambda_i = \lambda \mathbb{1}_{\{i \le k\}}$, $\mu_i = i\mu \mathbb{1}_{\{i \le k\}}$ $(\rho = \frac{\lambda}{\mu})$ M/M/k/k queue



$$Z = \sum_{i=0}^{k} \frac{\rho^i}{i!} < \infty \ \forall \rho, k, \quad \pi(x) = \frac{\rho^x}{Z\, x!} \ \forall x \le k$$

iv) $\lambda_i = \lambda$, $\mu_i = i\mu \mathbb{1}_{\{i>0\}}$ $(\rho = \lambda/\mu)$ M/M/∞ queue



$$Z = \sum_{i=0}^{\infty} \frac{\rho^i}{i!} = e^{\rho} < \infty \ \forall \rho$$

$$\pi(x) = \frac{e^{-\rho} \rho^x}{x!} = \mathrm{Poi}(\rho)$$

Reversibility

Birth-death chains are a special case of reversible CTMCs. To define these, consider any $T>0$, and CTMC $X(t)$. Then the time reversed process $\tilde{X}(t) = X(T-t)$ is a CTMC on $[0,T]$ with semigroup $\tilde{P}$ satisfying $\forall x,y \in \mathcal{X}$, the detailed balance eqn $\pi(x) P(x,y) = \pi(y) \tilde{P}(y,x)$. To avoid dependence on T, we extend $X(t)$ to negative time by defining $\{X(-t); t \geq 0\}$ as a CTMC with semigroup $\tilde{P}$.

Thm (CTMC Kelly's Lemma) - Let $X(t)$ be a regular DTMC with generator Q, and consider any dist π. Let $\tilde{Q}$ be defined such that $\pi(x) Q(x,y) = \pi(y) \tilde{Q}(y,x)$ $\forall x,y \in \mathcal{X}$, and $\tilde{Q}(x,x) = -\sum_{y \neq x} \tilde{Q}(x,y)$. If $\tilde{Q}(x,x) = Q(x,x)$ $\forall x \in \mathcal{X}$, then π is the stationary distr of Q, and $\tilde{Q}$ generates the reverse-time process $\tilde{X}(t)$.

- The proof is similar to the DTMC case. Moreover, if $\tilde{Q} = Q$, then $X(t)$ is reversible (ie., $X(t) \stackrel{d}{=} \tilde{X}(t)$)
- Corr: A stationary birth-death process is reversible.

Pf. Recall $\pi(x) = \frac{1}{Z} \cdot \frac{\lambda_0}{\mu_1} \cdot \frac{\lambda_1}{\mu_2} \cdots \frac{\lambda_{x-1}}{\mu_x}$. Thus for any $x \geq 1$, we have $\pi(x) \mu_x = \pi(x-1) \lambda_{x-1}$, and thus its reversible.

- A more surprising and useful consequence of reversibility occurs when we consider birth-death chains where $\lambda_x = \lambda \ \forall x \geq 0$, i.e., all the birth rates are the same. This can be interpreted as saying that the births follow a $PP(\lambda)$ process, independent of the state. Assume also that the chain is ergodic, i.e., $\sum_{i=1}^{\infty} \frac{\lambda^i}{\mu_1 \mu_2 \cdots \mu_i} < \infty$ (from ⊛)

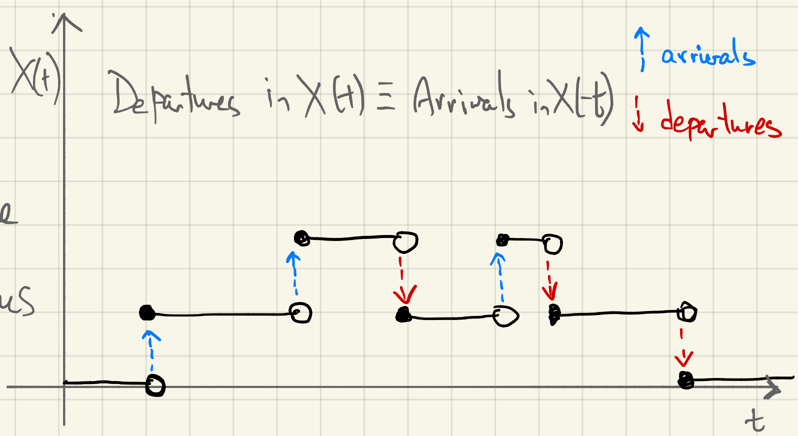
Thm (Burke's Thm) - Let $X(t)$ be a birth-death process with birth-rate $\lambda_x = \lambda \ \forall x \geq 0$, and let $A(s,t]$ and $D(s,t]$ denote the number of births (ie, arrivals) and deaths (ie, departures) in any interval $(s,t]$. Then

i) $\forall t$, $\{D(s,t], s<t\} \perp\!\!\!\perp X(t) \perp\!\!\!\perp \{A(t,u], u>t\}$

ii) The departure process is $PP(\lambda)$

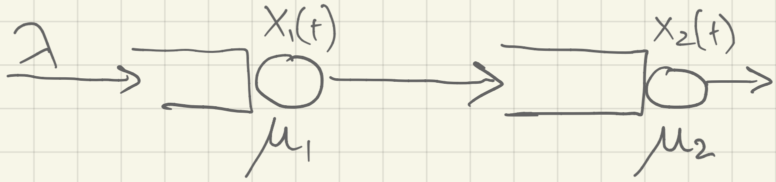
Pf - By reversibility, $X(t) \stackrel{d}{=} X(-t)$. Also the upward jumps in $X(t)$ (ie, arrivals) form a $PP(\lambda)$, and are equal in dist^n to the upward jumps in $X(-t)$, which correspond to departures in $X(t)$

- Also by construction, $A(t,w] \perp\!\!\!\perp X(t)$ for any $0 \leq t < w$. These however are departures $D(-w,-t]$ for $X(-t)$. Thus past departures $\perp\!\!\!\perp X(t)$



- This now allows us to build complex networks of queues!

Eg (Tandem Queues) - 2 M/M/1 queues is series



$\mathcal{X} \equiv \mathbb{N}_0 \times \mathbb{N}_0$

- Suppose $\rho_1 = \frac{\lambda}{\mu_1} < 1 \Rightarrow X_1(t) \sim M/M/1$, $\lim_{t\to\infty} \mathbb{P}[X_1(t)=x] = \rho_1^x(1-\rho_1)$

- By Burke's Thm, departures from 1 = arrivals from 2 $\sim PP(\lambda)$

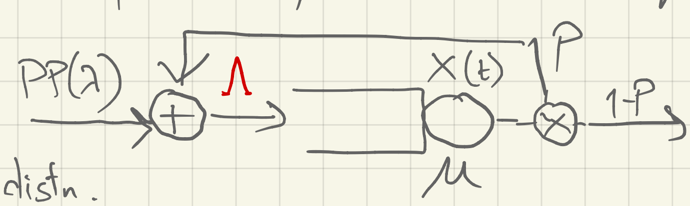
  Now if $\rho_2 = \frac{\lambda}{\mu_2} < 1$, then $X_2(t) \sim M/M/1$, $\lim_{t\to\infty} \mathbb{P}[X_2(t)=x] = \rho_2^x(1-\rho_2)$

- Claim - Stationary dist of $(X_1, X_2) \equiv \pi(x_1,x_2) = \rho_1^{x_1}\rho_2^{x_2}(1-\rho_1)(1-\rho_2)$

  Pf - Kelly's Lemma! Check $\pi(x_1,x_2)\,Q((x_1,x_2),(y_1,y_2)) = \pi(y_1,y_2)\,Q((y_1,y_2),(x_1,x_2))$

- Note - This does not mean $X_1(t) \perp\!\!\!\perp X_2(t)$! Rather, what it says is that they are independent under the stationary distⁿ π. Such a distribution is said to be product form.

Eg (Queue with Feedback) - Suppose departures from an M/M/1 queue returns w.p. p.



Suppose $X(t)$ converges to a stationary distn.

Then the 'steady-state' rate of arrivals Λ must obey $\Lambda = \lambda + p\Lambda \Rightarrow \Lambda = \frac{\lambda}{1-p}$

Now assume $\rho = \frac{\lambda}{\mu(1-p)} < 1$. Intuitively, $\pi(x) = (1-\rho)\rho^x \;\; \forall x \geq 0$

Again this is true - check by verifying reversibility!