ORIE 6180: Design of Online Marketplaces **Spring 2016**

Lectures 8 - 9 — February 29 & March 2

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8.1 Overview of the last lecture

In the last lecture we covered the Myerson - Satterthwaite theorem, which states that there is no mechanism that satisfies all following properties at the same time: individual rationality, balanced budget, incentive compatibility and efficiency (maximizing welfare).

8.2 Overview of this lecture

Following the results of [1], we are considering the case of auctions with interdependent valuations. We develop a model for such auctions using the notion of affiliated signals and then turn to the main result, Linkage Principle theorem, which informally states that in order to maximize revenue auctioneer should disclose all available information to bidders. A good source of reference is chapters 6 and 7 from [2].

8.3 Introduction

In previous lectures we analyzed auctions under following assumptions:

- Risk neutral agents
- Private valuations: value v_i does not depend on values of other players.
- Independent signals: $v_i \simeq F_i,\ F_i$ are independent

In today's lecture we are considering a more general setting, where the last two assumptions do not hold.

Note: The case of risk-averse/risk-seeking agents turns out to be considerably different (in particular, the Revenue Equivalence theorem does not apply in this case).

8.4 Mineral rights model

Before considering the general model, we start by looking at the "mineral rights" auction to develop intuition for auctions with signals and interdependent valuations. The framework for this auction is:

- Single item, common valuation V.
- Agent i has a signal X_i with following properties:
  1. $\mathbb{E}[X_i|V = v] = v$
  2. $X_i \perp\!\!\!\perp X_j \mid V$.

Example: One signal might be $X_i = v + \epsilon_i$, $\mathbb{E}[\epsilon_i] = 0$.
Consider running a second price auction in this setting. Note that

$$\mathbb{E}[\max\ X_i|V = v] \geq \max_i \mathbb{E}[X_i|\ V = v] = 0,$$

so the winning bidder will tend to overbid if he is bidding his signal. This phenomena is called the "winner's curse".

Another important thing to notice is that second price and ascending price auctions are no longer equivalent in this setting, they have different equilibria and revenue.

8.5 General model

- n risk-neutral bidders
- Bidder i gets signal X_i, auctioneer gets signal S

  Value of bidder i is determined by some function u: $v_i(X_i, X_{-i}, S) = u(X_i, X_{-i}, S)$
- v_i are nonnegative and continuous,

  $v(0, 0, 0) = 0$, $\mathbb{E}[v_i] < \alpha$
- v_i is nondecreasing in all variables, strictly in the X_i
- v_i is twice continuously differentiable
- $v_i(X_i, X_{-i}, S) = u(X_i, X_{-i}, S)$ (Symmetric valuations)

  Symmetric signals: u is invariant to 'shuffling' X_{-i}
- Symmetric F: X are drawn from the same distribution F, but are not necessarily independent. $S \sim F_s$
- Signals are affiliated

8.5.1 Affiliated signals and their properties

Definition 1. *Suppose X and X' are signal vectors drawn from distribution with density f. We say that signals are affiliated if*

$$f(X \vee X')f(X \wedge X') \geq f(X)f(X')$$

Intuitively, this condition means that if some signal happened to be large then the other signals are more likely to be large too.

We now list two important properties of affiliated signals:

1. Let signals be $(X_1, X_2, \ldots, X_n)$. Let $(Y_1, Y_2, \ldots, Y_{n-1})$ be sorted order of $(X_2, X_3, \ldots, X_n)$. Then

$$\{X_i\} \text{ is affiliated} \quad \Longrightarrow \quad (X, Y_1, \ldots, Y_{n-1}) \text{ is affiliated.}$$

2. Suppose γ is a non-decreasing function and $\{X_i\}$ is affiliated, and that $x' > x$. Then

$$\mathbb{E}[\gamma(Y_1)|x_1 = x'] \geq \mathbb{E}[\gamma(Y_1)|x_1 = x].$$

8.5.2 Second price auction result

We define $v(x, y) = \mathbb{E}[v_1|X_1 = x, Y_1 = y] = \mathbb{E}[u(X_1, Y_1, \ldots, Y_{n-1}, S)|X_1 = x, Y_1 = y]$

Theorem 8.1. *If $v(x, y)$ is strictly increasing in x and non-decreasing in y, then $\beta^{II}(x) = v(x, x)$ is a symmetric equilibrium in a 2^{nd} price auction. (So every player coming with value x bids $v(x, x)$.)*

Proof: Consider bidder 1 without loss of generality, and assume that $i \neq 1$ plays $\beta^{ii}(x_i)$. The payoff if $i = 1$ plays b is:

$$\pi(b, x) = \int_0^{\beta^{-1}(b)} (v(x, y) - v(y, y))g(y|x) \, \mathrm{d}y.$$

Note that $v(x, y)$ is increasing in x, so $v(x, y) - v(y, y) \geq 0 \iff x \geq y$. π is maximized if $\beta^{-1}(b) = x$, and hence $b = v(x, x) = \beta^{II}(x)$. □

8.6 The Linkage Principle

Let A be a standard auction, where a single item is awarded to the highest bidder. Suppose auction A has a symmetric, increasing equilibrium β^A where all bidders have the same strategy. We consider bidder 1, assuming that all other bidders bid according to the equilibrium strategy β^A. Let $W^A(z, x)$ denote the expected payment by bidder 1 if she receives the signal $X_1 = x$, bids according to $\beta^A(z)$ (as if she received signal z), and wins.

Example: Consider a first price auction, denoted I. Under such an auction, and su pposing that player 1 wins, she is asked to pay her bid. Under the assumed equilibrium strategy, thi s is $\beta^I(z)$. Thus $W^I(z, x) = \beta^I(z)$. In a second price auction II, assuming player 1 wins, she pays according to t he bidder who received signal Y_1, and since player 1 won, $Y_1 < z$. Hence,

$$W^{II}(z, x) = \mathbb{E}[\beta^{II}(Y_1)|X_1 = x, Y_1 < z].$$

In general, for an auction A, we define $W_2^A(z, x) = \frac{\partial W^A(z,x)}{\partial x}|_{(z,z)}$.

Theorem 8.2 (Linkage Principle). *Let A, B be two standard auctions with respective symmetric increasing equilibrium β^A, β^B. Suppose that only the winner pay a positive amount in both auctions, and that*

1. $W^A(0, 0) = W^B(0, 0) = 0$,
2. *For all x, $W_2^A(x, x) \geq W_2^B(x, x)$.*

Then $W^A(x, x) \geq W^B(x, x)$.

Since both auctions earn revenue only though the payment of the highest bidder, this principal means that the revenue of auction A is higher. In words, this principle means that "the more closely the winning bidder's payment is linked to his signal, the higher the expected revenue is."

Example, ctd: In the first price auction,

$$W_2^I(z, x) = \frac{\partial}{\partial x} W^I(z, x)|_{(z,z)} = \frac{\partial}{\partial x} \beta^I(z)|_{(z,z)} = 0.$$

In the second price auction, $\beta^{II}(x) = v(x, x) = \mathbb{E}[V_1|X_1 = x, Y_1 = x]$. Thus:

$$W^{II}(z, x) = \mathbb{E}[v(Y_1, Y_1)|Y_1 < z, X_1 = x].$$

Since Y_1, X are affiliated and $v(x, y)$ is nondecreasing in x, y, so for all x, z

$$W_2^{II}(z, x) = \frac{\partial}{\partial x} \mathbb{E}[v(Y_1, Y_1)|Y_1 < z, X_1 = x]|_{(z,z)} \geq 0.$$

From the linkage principal, we can conclude the the expected revenue of the second price auction is at least as large as that of the first price auction. That is, a direct consequence of the linkage principal is that, under affiliated valuations,

$$\mathbb{E}[R^{II}] \geq \mathbb{E}[R^I],$$

where R^A denotes the revenue in auction A.

We can also obtain the revenue equivalence formula as a corollary of the linkage principal, which may be on the second homework; see Proposition 7.8 in [3].

In addition, releasing the auctioneer signal S leads to greater expected revenue. Let $W^I(z, x, s) := \beta^I(z, s)$, where everyone participant in the auction has signal S. If the information in S is affiliated, then S is in some way correlated with X. We claim that $\beta^I(z, s)$ exists and is non-decreasing, which implies that $W_2^I(z, x, s) \geq 0$. See section 7.13.3 of [3]. Let $A \in \{AP, SP, FP\}$ be an ascending price auction, second price auction, or first price auction. Then the linkage principal also implies that the expected revenue of auction A with signal S made public is at least as large as the expected revenue with signal S withheld. Thus, if a seller believes in the purview of the linkage principle, that seller should release all information they have. For example, consider an art auction. If the auctioneer gets an independent art expert to evaluate the painting, the principle suggests that the auctioneer should publicly release that report.

Putting everything together so far, we have the following hierarchy:

$$\begin{array}{ccccc} AP \text{ with } S \text{ released} & \geq & SP \text{ with } S \text{ released} & \geq & FP \text{ with } S \text{ released} \\ \vee\!| & & \vee\!| & & \vee\!| \\ AP \text{ without } S \text{ released} & \geq & SP \text{ without } S \text{ released} & \geq & FP \text{ without } S \text{ released.} \end{array}$$

8.7 A Quick Useful Lemma

Lemma 8.3. *Suppose g, h are continuous differentiable functions such that $g(0) \geq h(0)$. If for every $x \geq 0$, $g(x) \leq h(x)$ implies $g'(x) \geq h'(x)$ then $g(x) \geq h(x)$ for every $x \geq 0$.*

Proof: (As given by Quint [4]) Suppose instead that $g(y) < h(y)$ for some $y > 0$. Let $z = \sup\{x < y \mid g(x) \geq h(x)\}$. By continuity, $g(z) = h(z)$ and $g(x) < h(x)$ for all $x \in (z, y]$, and recall now that by assumption $g'(x) \geq h'(x)$ in that interval. Then,

$$g(y) - g(z) = \int_z^y g'(x)\ \mathrm{d}x \geq \int_z^y h'(x)\ \mathrm{d}x = h(y) - h(z)$$

Since $g(z) = h(z)$, this is simply

$$g(y) \geq h(y)$$

and so we have proved the lemma by contradiction, since we assumed $g(y) < h(y)$. □

8.8 Proof of the Linkage Principle

Proof: Let $\Delta(x) = W^A(x, x) - W^B(x, x)$. Then,

$$\Delta'(x) = (W_1^A(x, x) - W_1^B(x, x)) + (W_2^A(x, x) - W_2^B(x, x)).$$

Consider auction A and assume that for $i \neq 1$ bidder i plays $\beta^A(x_i)$. Let $G(z \mid x) = P(Y < z \mid X_1 = x)$. This implies that bidder 1 maximizes

$$\gamma(z, x) := \int_0^z v(x, y) g(y \mid x)\ \mathrm{d}y - G(z, x) W^A(z, x).$$

If β^A is an equilibrium then we have a first order condition. Now, using Leibniz rule and the fact that $\frac{\partial \gamma(z,x)}{\partial z}\Big|_{z=x} = 0$ (due to the maximization assumption), we see

$$0 = g(x \mid x)v(x,x) - g(x \mid x)W^A(x,x) - G(x \mid x)W^A(x,x),$$

so that

$$W_1^A(x,x) - W_1^B(x,x) = \frac{-g(x,x)}{G(x \mid x)}(W^A(x,x) - W^B(x,x)).$$

Recalling the definition of $\Delta(x)$ and its derivative, we see that this is

$$\Delta'(x) - (W_2^A(x,x) - W_2^B(x,x)) = \frac{-g(x,x)}{G(x \mid x)}\Delta(x)$$

meaning that

$$\Delta'(x) = \frac{-g(x,x)}{G(x \mid x)}\Delta(x) + (W_2^A(x,x) - W_2^B(x,x)).$$

□

Note now that the previously mentioned revenue rankings all follow directly from this result, as mentioned at the end of Section 8.6.

8.9 A Review and a Preview

Up to now in this class we've dealt with mechanisms that have private valuations. This includes topics like DSIC, VCG & welfare maximization, Bayes-Nash equilibria, the Myerson Lemma, revenue optimization auctions involving virtual welfare, simple approximation auctions, Myerson-Satterthwaite, and the Linkage Principle. Next we are going to cover topics in Price Theory. This involves us exposing strategic behavior to external factors through demand functions such as $d(p) = 1 - F_v(p)$. Caillard & Julliene, Rochet & Triole, Armstrong, and Wey all provide good papers in the subject.

Bibliography

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