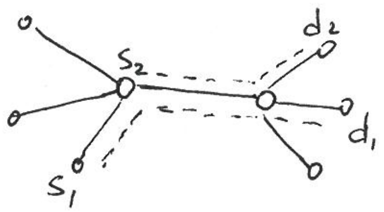
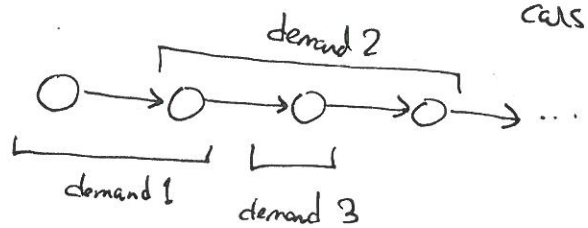
Network Revenue Management

- Capacity control across multiple resources linked by demands for sets of resources

- Eg - Hub and spoke networks (airlines)



Linear networks (hotels, rental cars)



- No natural ordering of fare-classes $\Rightarrow$ Use time for DP formulation

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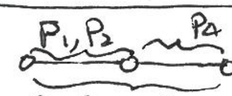
Problem setting

- m resources, n products (or ODFs $\equiv$ Origin - Dest^n - Fare)
- $\underline{c} = (c_1, c_2, \ldots, c_m) \in \mathbb{Z}^m$
- $t \equiv$ time-to-go $\in [T, 0]$
- demand for product $i \sim PP(\lambda_i(t))$
- product $i \equiv$ requirements $\underline{a}_i \in \{0,1\}^n$, price p_i
- State - $(t, \underline{x})$
- Bellman Eqn - $V(t, 0) = V(0, \underline{x}) = 0$

Eg -



$$A = \begin{array}{c} \begin{array}{cccc} a_1 & a_2 & a_3 & a_4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{array} \quad P_3$$

$p = (p_1 \;\; p_2 \;\; p_3 \;\; p_4)$

$$\frac{\partial V(t, \underline{x})}{\partial t} = \sum_{i=1}^{n} \lambda_i(t) \max_{u_i \in \{0,1\}} \Big[p_i u_i - \underbrace{\Delta_i V(t, \underline{x})}_{= V(t,\underline{x}) - V(t, \underline{x} - \underline{a}_i)} \Big]^+$$

$$\text{s.t.} \quad \underline{x} - \underline{a}_i \geq 0$$

- Let $R(\underline{\theta}) = \max_{\underline{u} \in \{0,1\}^n} \sum_{i=1}^{n} \lambda_i(t) [P_i u_i - \theta_i]^+$

Then $\underline{u}^* \equiv \{ u_i = 1 \text{ if } P_i \geq \theta_i \} \Rightarrow R(\underline{\theta}) = \sum_{i=1}^{n} \lambda_i(t) [P_i - \theta_i]^+$

- $$\frac{\partial V(t,\underline{x})}{\partial t} = \sum_{i=1}^{n} \lambda_i(t) \left[R(\Delta V(t,\underline{x})) \right]$$

and $\underline{u}^*(t,\underline{x}) = \{ \underline{u_i = 1 \text{ iff } \underline{x} - \underline{a}_i \geq 0 \text{ and } P_i \geq \Delta_i V(t,\underline{x})} \}$

- Difficulty -
  i) $V(t,x)$ may be difficult to compute
  ii) $\underline{u}^*(t,x)$ needs too much storage / is complicated

We want simpler policies with good properties

- <u>Idea</u> - <u>bid-price controls</u> / <u>Probabilistic admission control</u>

Suppose
$$\Delta_k V(t,\underline{x}) = V(t,\underline{x}) - V(t, \underline{x} - \underline{a}_i)$$
$$\approx \sum_{j=1}^{m} \underbrace{\frac{\partial V(t,\underline{x})}{\partial x_j}}_{\triangleq\ \mu_j(t)} a_i(j)$$

$\triangleq \mu_j(t)$ - bid-price for resource j at time t

(This seems likely when $\underline{a}_k \ll \underline{x}$)

Now condition for admission $\equiv$ $P_i \geq \sum_{j=1}^{m} \mu_j(t) a_i(j)$

(and $\underline{x} - \underline{a}_i \geq 0$)

- First, we rescale time to get a discrete time equivalent to the Bellman eqn
  - Given $a > 1$, we set $T \to aT$, $\lambda_j(t) \to \frac{\lambda_j(at)}{a}$

    and consider discrete times $\{Ta, aT-1, \dots, 0\}$

    (henceforth, we use T and $\lambda_j(t)$ to refer to scaled values)
  - a is chosen s.t $\lambda_j(at)/a \ll 1 \quad \forall\, t$

Now we have $\boxed{V(t,\underline{x}) - V(t-1,\underline{x}) = R\left(\Delta V(t-1,\underline{x})\right)}$

(where again $R(\underline{\theta}) = \sum_{i=1}^{n} \lambda_i(t)\,[p_i - \theta_i]^+$, $V(t,0) = V(0,x) = 0$

and opt control $u_i^*(t,x) = \mathbb{1}\{\underline{a}_i \le x \text{ AND } p_i \ge \Delta_i V(t,x)\}$)

- To implement bid-prices, we need linear (in $\underline{x}$) approx^n of $V(t,x)$

(Fluid) upper bound on $V(t,x)$

- Oracle based bound: Suppose we know realization of demand
  - Let $D_j \sim \mathrm{Poi}\left(\int_0^T \lambda_j(s)\,ds\right)$ be total arrivals to class j (ie, total demand for product j). Define $\Lambda_j = \int_0^T \lambda_j(s)\,ds$

- $$V^U(T,\underline{c}\,|\,D) \equiv \max \sum_{i=1}^{n} p_i y_i$$

  (*)
  $$\text{s.t} \quad \sum_{i=1}^{n} a_i(j)\, y_i \le c_j \quad \forall j \in [M]$$
  $$0 \le y_i \le D_i \quad \forall i \in [N]$$

Claim - $V^U(T, \underline{c} \mid D)$ is concave in D

Pf - If y and y' are solns to $V^U(T, \underline{c} \mid D)$ and $V^U(T, \underline{c} \mid D')$, then $\alpha y + (1-\alpha) y'$ is feasible for $\alpha D + D'(1-\alpha)$

$$\Rightarrow V^U(T, \underline{c} \mid D\alpha + D'(1-\alpha)) \geq \alpha V^U(T, \underline{c} \mid D) + (1-\alpha) V^U(T, \underline{c} \mid D')$$

- Thus, if we replace D_j with $\mathbb{E}[D_j] = \Lambda_j$ in (*), we get $V^{fluid}(T, \underline{c})$, which by Jensen's satisfies $V^{fluid}(T, \underline{c}) \geq \mathbb{E}[V^U(T, \underline{c} \mid D)]$

- Primal-Dual forms for fluid problem $V^{fluid}(T, \underline{c})$

Primal

$$\max_{y_i} \sum_{i=1}^{n} p_i y_i$$

dual vars

$$\text{s.t} \quad \sum_{i=1}^{n} a_i(j)\, y_i \leq c_j \quad \forall j \in [M] \qquad z_j$$

$$0 \leq y_i \leq \Lambda_i \quad \forall i \in [N] \qquad \beta_i$$

Dual

$$\min \sum_{i=1}^{n} \Lambda_i \beta_i + \sum_{j=1}^{m} c_j z_j$$

$$\text{s.t} \quad \sum_{j=1}^{m} a_i(j)\, z_j + \beta_i \geq p_i \quad \forall i \in [N]$$

$$z_j \geq 0, \quad \beta_i \geq 0$$

- By complementary slackness

$$z_j^* > 0 \Rightarrow \sum_{i=1}^{n} a_i(j)\, y_i = c_j$$

$$\beta_i^* > 0 \Rightarrow \quad y_i = \Lambda_i$$

$z_j^* \equiv$ 'marginal cost' of unit of resource j

$\beta_i^* \equiv$ 'marginal cost' of additional customer for product i

- $$\beta_i^* = \left(P_i - \sum_{j=1}^{m} a_i(j) z_j^*\right)^+ \quad \forall i$$

$$\Rightarrow \sum_{i=1}^{n} \Lambda_i \beta_i^* = \sum_{i=1}^{n} \Lambda_i \left(P_i - \sum_{j=1}^{m} a_i(j) z_j^*\right)^+ = R\left(A^T \underline{z}^*\right)$$

$$\Rightarrow \quad V^{fluid}(T, c) = \min_{z \geq 0} \left\{ R(A^T \underline{z}) + c^T \underline{z} \right\}$$

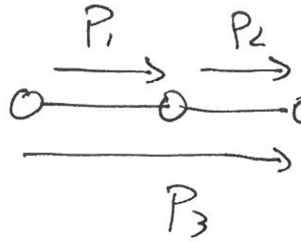
(Equivalently - we are approximating $\Delta_i V(T,c) \approx \sum_{j=1}^{m} a_i(j) z_j^*$)

- Now given y_i^* (primal opt vars) or z_j^* (dual opt vars), we have 2 simple heuristics
  - i) Bid price $\equiv$ Admit i if $\sum_{j=1}^{m} a_i(j) z_j^* \leq P_i$ and $\underline{x} - \underline{a}_i \geq 0$
  - ii) Probabilistic admission control (PAC) $\equiv$ Admit i wp $\frac{y_i^*}{\Lambda_i}$ if $\underline{x} - \underline{a}_i \geq 0$

- The notion of a bid price is more general. Given any linear approx^n of $\Delta_k V(t, \underline{x}) \approx \sum_{j \in [M]} \mu_j a_i(j)$, μ_j are bid prices
(Similarly for PAC policies)

- Are bid-prices optimal? No

Eg -



- $P_1 = P_2 = 250$, $P_3 = 450$
- $T = 2$

Suppose $\underline{\lambda}(2) = (0.3, 0.3, 0.4)$

$\underline{\lambda}(1) = (0, 0, 0.8)$ (and no arrival wp 0.2)

<u>Claim</u> - OPT $\equiv$ Accept only customers for product 3

$\Rightarrow R^* = (0.4 + 0.6 \times 0.8) \times 450 = 396$

However, to implement this via bid price μ_i, we need

$\mu_1 + \mu_2 \leq 450, \; \mu_1 > 250, \; \mu_2 \geq 250 \Rightarrow$ impossible

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- Can we show bid-prices perform well? Yes!

<u>Idea</u> - Consider the problem under a 'large market' scaling

$c \to \theta c \;,\; \lambda_i(t) \to \theta \lambda_i(t) \;\; \forall i, t$ for some $\theta > 1$

- Note $\{Z_j^*\}$ remains the same

$$\left(V^{\theta, fluid}(T, c) = \max \; \theta \left(\sum_{i=1}^{n} \Lambda_i \beta_i + \sum_{j=1}^{m} c_j z_j \right) \quad \text{s.t.} \; \sum_{j \in [M]} a_i(j) z_j + \beta_i \geq p_i \; \forall i, \;\; z_j, \beta_i \geq 0 \right)$$

- Let $V^\theta(T, c)$ be the value fn under θ scaling

We know $V^\theta(T,c) \leq \mathbb{E}\left[V^{\theta, H}(T, c \mid D)\right] \leq V^{\theta, fluid}(T, c)$

We first need an additional Lemma

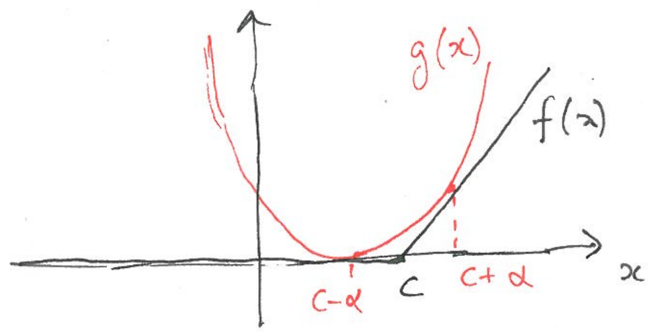
Lemma - For any r.v. X with $\mathbb{E}[X] = \mu$, $\mathrm{Var}(X) = \sigma^2$, $\forall c$

$$\mathbb{E}\left[(X-c)^+\right] \leq \frac{1}{2}\left(\sqrt{\sigma^2 + (c-\mu)^2} - (c-\mu)\right)$$

Pf Consider $f(X) = (X-c)^+$

Moreover, for any $\alpha > 0$, define

$$g(x) = \frac{(x-(c-\alpha))^2}{4\alpha}$$



Then (from figure) we have $g(x) \geq f(x)\ \forall x$

$$\Rightarrow \mathbb{E}[f(X)] \leq \mathbb{E}[g(X)] = \frac{1}{4\alpha}\mathbb{E}\left[X^2 - 2X(c-\alpha) + (c-\alpha)^2\right]$$

$$\Rightarrow \mathbb{E}\left[(X-c)^+\right] \leq \min_{\alpha \geq 0}\left(\frac{\mu^2 + \sigma^2 - 2\mu(c-\alpha) + (c-\alpha)^2}{4\alpha}\right)$$

Setting $\alpha = \sqrt{\sigma^2 + (c-\mu)^2}$, we get

$$\mathbb{E}\left[(X-c)^+\right] \leq \frac{\sqrt{\sigma^2 + (c-\mu)^2} - (c-\mu)}{2}$$

Note: $\mathbb{E}\left[(X-c)^+\right] \leq 0.5\sigma + 0.5\left(|c-\mu| - (c-\mu)\right) = 0.5\sigma$ if $c \geq \mu$

Thm (Talluri & Van Ryzin '98) - Let B^θ be the total expected revenue under the bid price heuristic using bid prices $\{z_j^*\}$ from the fluid LP. Then

$$\frac{B^\theta}{V^\theta(T,c)} \geq 1 - O\left(\frac{1}{\sqrt{\theta}}\right)$$

(strictly speaking - this requires a small modification to the policy - see below)

Pf - We consider a small modification of the basic bid-price heuristic, as follows (based on Reiman & Wang '07)

Recall for $V^{\theta,\text{fluid}}(T,c)$, the primal soln is $\{y_i^*\theta\}_{i\in[N]}$, and dual soln is $\{z_j^*\}_{j\in[M]}$. Now consider the following policy

- If $p_i > \sum_{j=1}^m a_i(j)\, z_j^*$ (and $\underline{x}-\underline{a}_i \geq 0$) : Admit i
- If $p_i = \sum_{j=1}^m a_i(j)\, z_j^*$ (and $\underline{x}-\underline{a}_i \geq 0$) : Admit i wp $\frac{y_i^*}{\Lambda_i}$
- Else reject i

Now we show that under this policy, revenue $\bar{B}^\theta$ satisfies

$$\bar{B}^\theta \geq \left(1 - O\left(\tfrac{1}{\sqrt{\theta}}\right)\right) V^\theta(T,c)$$

Now we have the following.

1) $\forall\, \theta$, $\quad V^{\theta, fluid}(T,c) = \theta\, V^{fluid}(T,c)$

$$V^{\theta}(T,c) \leq V^{\theta, fluid}(T,c)$$

$$\Rightarrow \frac{\bar{B}^{\theta}}{V^{\theta}(T,c)} \geq \frac{\bar{B}^{\theta}}{\theta V^{fluid}(T,c)}$$

(Note: by Jensen's $\bar{B}^{\theta} \leq \theta\, V^{fluid}(T,c)$)

2) We can write $y_i^* = \Lambda_i \cdot \left(\frac{y_i^*}{\Lambda_i}\right) \quad \forall i$

$\Rightarrow$ The fluid LP $\equiv$ $V^{fluid}_{(T,c)} = \sum_{i\in[n]} \Lambda_i \left(\frac{y_i^*}{\Lambda_i}\right) P_i$

and also (by feasibility) $\quad \sum_{i\in[n]} \Lambda_i \left(\frac{y_i^*}{\Lambda_i}\right) \cdot a_i(j) \leq c_j \quad \forall j$

3) Now consider an alternate admission policy, where we admit <u>all</u> arriving customers in class i w.p $\frac{y_i^*}{\Lambda_i}$, ignoring capacity constraints; however, we incur a cost of $P_{max} \triangleq \max_{i\in[n]} \{P_i\}$ for each additional unit of capacity used on any leg

Let $L^\theta \equiv$ revenue under this new policy

$$\Rightarrow \quad \text{i)} \quad L^\theta = \sum_{i\in[n]} \theta \Lambda_i \left(\frac{y_i^*}{\Lambda_i}\right) p_i - \underbrace{\mathbb{E}\left[\sum_{j=1}^{m} p_{max}\left(\sum_{i\in[n]} D_i^\theta\left(\frac{y_i^*}{\Lambda_i}\right) a_i(j) - \theta c_j\right)^+\right]}_{\text{Overbooking cost} \ \triangleq\ C_{over}}$$

$$= \theta\left(\sum_{i\in[n]} \Lambda_i \left(\frac{y_i^*}{\Lambda_i}\right) p_i\right) - C_{over}$$

$$= \theta\, V^{fluid}\left(1 - \frac{C_{over}}{\theta V^{fluid}}\right)$$

ii) $L^\theta \leq \bar{B}^\theta$ <u>sample-pathwise</u>

This is because admitting a customer when capacity is unavailable costs more than revenue earned

- Combining the 4 inequalities, we get

$$\frac{\bar{B}^\theta}{V^\theta(T,c)} \geq 1 - \frac{C_{over}}{\theta V^{fluid}}$$

where $V^{fluid} = \left(\sum_{i\in[n]} y_i^* p_i\right)$

and $C_{over} = p_{max} \sum_{j\in[m]} \mathbb{E}\left[\left(\sum_{i\in[n]} D_i^\theta\left(\frac{y_i^*}{\Lambda_i}\right) a_i(j) - \theta c_j\right)^+\right]$

Finally - $\mathbb{E}\left[\sum_{i\in[n]} D_i^{\theta}\left(\frac{y_i^*}{\Lambda_i}\right) a_i(j)\right] = \sum_{i\in[n]} \theta y_i^* a_i(j)$

- $\theta c_j \geqslant \theta \sum_{i\in[n]} y_i^* a_i(j)$

$\Rightarrow$ We can use $\mathbb{E}\left[(X-c)^+\right] \leq 0.5\,\sigma$

- $\sqrt{\mathrm{Var}\left(\sum_{i\in[n]} D_i^{\theta}\left(\frac{y_i^*}{\Lambda_i}\right) a_i(j)\right)} \leq \sqrt{\theta\left(\sum_{i\in[n]} a_i(j) y_i^*\right)}$

$$\Rightarrow \quad \frac{\bar{B}^{\theta}}{V^{\theta}(T,c)} \geqslant 1 - \frac{P_{max}\sqrt{\theta}\sqrt{\sum_{i\in[n]} a_i(j) y_i^*}}{2\left(\sum_{i\in[n]} y_i^* p_i\right)\theta}$$

$$= 1 - O\left(\frac{1}{\sqrt{\theta}}\right)$$