Multi fare-class capacity allocation (contnd.)

- **Problem** - Capacity c, fare classes $\underset{(D_1,F_1)}{P_1} \le \underset{(D_2,F_2)}{P_2} \le \ldots \le \underset{(D_n,F_n)}{P_n}$

- Bellman Eqn - If control $\equiv$ # of seats allocated

$$V_j(s) = \max_{y \in \{0,1,\ldots,s\}} \mathbb{E}_{F_j}\left[P_j \min(D_j, y) + V_{j+1}(\max(s-y, s-D_j))\right]$$

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2 approaches to solving -

1) When D_j, y are continuous

- Assume D_j available before taking action

$$\bar{V}_j(s) = \mathbb{E}_{F_j}\Big[\max_{y \in \underbrace{\{(s-D_j)^+, \ldots, s\}}_{\text{random set } Y(s, D_j)}} \{P_j(s-y_j) + \bar{V}_{j+1}(y)\}\Big]$$

- Show that $\bar{V}_j(s)$ is concave in y (TODO)

- Consequently, $y^*(s) = \begin{cases} s - \min\{x_j^*, D_j\} & ; \ s \geq x_j^* \\ s & ; \ s < x_j^* \end{cases}$

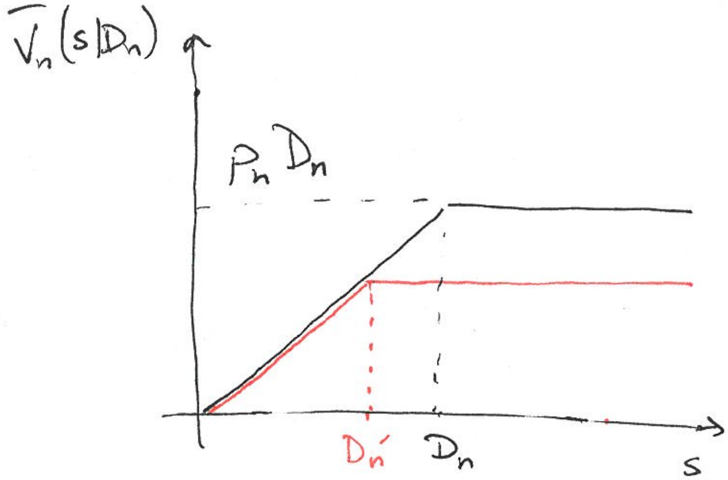
where $x_j^* = \arg\max_x \bar{V}_{j+1}(x)$

2) We study monotonicity properties of $\bar{V}_j(s)$ w.r.t s and j (for discrete capacity allocations)

Concavity of $\bar{V}_j(\,)$

- Induction on $j \in \{n, n-1, \ldots, 1\}$

- $$\bar{V}_n(s \mid D_n) \triangleq \max_{(s-D_n)^+ \le y \le s} \Big[\underbrace{-p_n(s-D_n)^+ + p_n s}_{\text{alt form for } p_n \min\{s, D_n\}}\Big]$$



- Random fn (depends on D_n)
- Concave in s $\forall D_n \in \mathbb{R}^+$
- $\bar{V}_n(s) = \mathbb{E}[\bar{V}_n(s \mid D_n)]$

  $\Rightarrow$ linear combination of concave fns

  $\Rightarrow$ concave in s

- Assume $\bar{V}_{j+1}(s)$ is concave in s,
  - Let $H_j(y) = [-p_j y + \bar{V}_{j+1}(y)]$ (concave in s)

    $$V_j(s \mid D_j) = \max_{(s-D_j)^+ \le y \le s} [H_j(y)] + p_j s$$

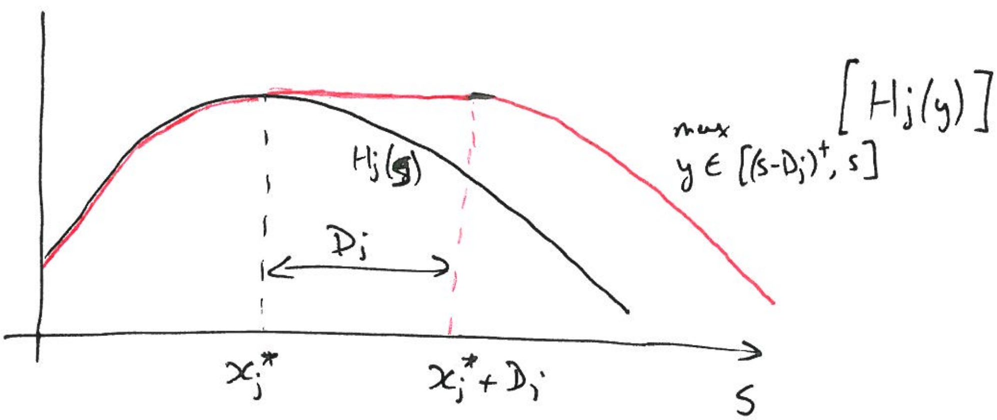
    If $\max_{(s-D_j)^+ \le y \le s} [H_j(y)]$ concave $\Rightarrow V_j(s) = \mathbb{E}[V_j(s \mid D_j)]$ is concave

  - Define $$\boxed{x_j^* = \arg\max_{y \in [0, \infty]} [H_j(y)]}$$

Note - $x_{n-1}^* = F_n^{-1}\left(1 - \frac{p_{n-1}}{p_n}\right)$ (Same as 2-class setting)

Now from previous discussion re maximizing a concave fn over a random interval, we have

$$y_j^* = \underset{y \in [(s-D_j)^+,\, s]}{\arg\max} \left[H_j(y) \right] = \begin{cases} s \; ; & s \le x_j^* \\ x_j^* \; ; & s - D_j \le x_j^* \le s \\ s - D_j \; ; & s - D_j \ge x_j^* \end{cases}$$



Thus

$$\max_{y \in [(s-D_j)^+),\, s]} \{ H_j(y) \}$$

is concave in s $\forall \; D_j$

- Note - the above argument also works if D_j is discrete

More over, we can modify it to show

$$\boxed{\Delta \bar{V}_j(s) = \bar{V}_j(s+1) - \bar{V}_j(s) \le \Delta \bar{V}_j(s-1)} \quad \forall s$$

(This is an intuitive 'diminishing returns' property of increasing capacity)

Alternate characterization of protection levels

- <u>Idea</u> - Show protection levels are <u>monotone</u>
  - Use it to get interpretable characterization of protection levels (including for discrete protection levels)

<u>Claim</u> ⊛ : $\bar{V}_j(s+1) - \bar{V}_j(s) \geqslant \bar{V}_{j+1}(s+1) - V_{j+1}(s) \quad \forall j, \forall s$

- Intuitive 'increasing returns' property - More capacity later has higher returns
- <u>Implication</u> - $\boxed{\text{Monotonicity of } x_j^*}$
  - Recall $x_j^* = \underset{y \in [0,\infty)}{\arg\max} \left[-P_j\, y + \bar{V}_{j+1}(y) \right]$

<u>Alt</u> : x_j^* is <u>first</u> y s.t $\underbrace{\bar{V}_j(y+1) - \bar{V}_j(y)}_{\triangleq \Delta \bar{V}_j(y)} < 0$

$$\Rightarrow \; -P_j + \bar{V}_j(x_j^*+1) - \bar{V}_j(x_j^*) < 0$$

$$\Rightarrow \; -P_{j+1} + \bar{V}_{j+1}(x_j^*+1) - \bar{V}_{j+1}(x_j^*) < 0 \quad \left(\text{from ⊛ and } P_{j+1} \geqslant P_j\right)$$

$$\Rightarrow \; \boxed{x_{j+1}^* \leqslant x_j^*} \quad \left(\text{Since } x_{j+1}^* \text{ is smallest } y \text{ where } \Delta V_{j+1}(y) < 0\right)$$

$\Rightarrow \; x_1^* \geqslant x_2^* \geqslant \dots \geqslant x_n^* = 0$ - NESTED Protection levels

Combining claim (*) with our previous 'diminishing returns of capacity' result, we get.

$\boxed{\begin{array}{l} \underline{\text{Thm}} - \quad \forall\, j \in \{1,2,\dots,n\text{-}1\},\ s \in \{0,1,\dots,c\} \\ \text{i)}\quad \Delta \bar{V}_j(s+1) \leq \Delta \bar{V}_j(s) \\ \text{ii)}\quad \Delta \bar{V}_j(s) \geq \Delta \bar{V}_{j+1}(s) \end{array}}$

<u>Pf</u> - We have already argued (i). For (ii), we consider 2 cases

<u>Case 1</u> - $s+1 \leq x_j^*$ ($\Rightarrow s \leq x_j^*$, i.e., capacity below x_j^*)

$$\Rightarrow \bar{V}_j(s \mid D_j) = V_{j+1}(s), \quad \bar{V}_j(1+s \mid D_j) = V_{j+1}(s+1) \ \forall D_j$$

$$\Rightarrow \Delta V_j(s) = \Delta \bar{V}_{j+1}(s)$$

<u>Case 2</u> - $s \geq x_j^*$ ($\Rightarrow s+1 \geq x_j^*$; We accept $\min\{s-x_j^*, D_j\}$)

$$\Rightarrow \Delta \bar{V}_j(s \mid D_j) = P_j\left[\min(s-x_j^*+1, D_j) - \min(s-x_j^*, D_j)\right] + V_{j+1}(\max(x_j^*, s+1-D_j)) - V_{j+1}(\max(x_j^*, s-D_j))$$

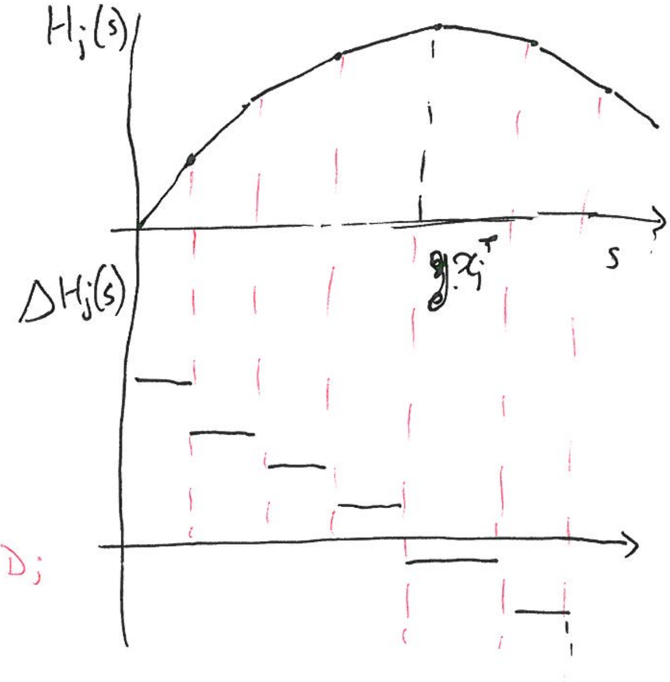
$$= P_j \cdot \mathbb{1}\{D_j \geq s+1-x_j^*\} + \mathbb{1}\{D_j \leq s-x_j^*\}\left[V_{j+1}(s+1-D_j) - V_{j+1}(s-D_j)\right]$$

$$= P_j + \mathbb{1}\{D_j \leq s-x_j^*\}\left[\Delta H_j(s-D_j)\right]$$

⑥

- Now from before, we know $H_j(s)$ is concave (and thus $\Delta H_j(s-D_j)$ is decreasing)

Moreover, $x_j^* \equiv \min\limits_{y \in [0,\infty]} \{ \Delta H_j(y) < 0 \}$



$\Rightarrow \mathbb{1}\{s \geq x_j^* + D_j\} \, \Delta H_j(s - D_j) \leq 0 \quad \forall D_j$

and also decreasing in s

$$\Rightarrow \mathbb{1}\{s \geq x_j^* + D_j\} \, \Delta H_j(s-D_j)$$

increasing $s \to s + D_j$

$$\geq \mathbb{1}\{s \geq x_j^*\} \, \Delta H_j(s)$$

$$= \mathbb{1}\{\underbrace{s \geq x_j^*}_{}\} \left(-P_j + V_{j+1}(s+1) - V_{j+1}(s) \right)$$

This is true ~~under~~ under our assumption

$$\Rightarrow \Delta \bar{V}_j(s \mid D_j) \geq V_{j+1}(s+1) - V_{j+1}(s) = \Delta V_{j+1}(s)$$

$$\Rightarrow \Delta V_j(s) \geq \Delta V_{j+1}(s) \qquad \blacksquare$$