Bounds for single-resource allocations via LP

- We want to use LPs to get bounds for the revenue under single-resource allocation
  - Setting

    C units, n classes $(D_n, p_n), (D_{n-1}, p_{n-1}), \ldots, (D_1, p_1)$
  - Idea - Suppose the demands arrive simultaneously

$$V_n^{UB}(c) \equiv \max \sum_{i=1}^{n} p_i x_i$$

(given $D_n, D_{n-1}, \ldots, D_1$)

$$\text{s.t} \quad \sum_{i=1}^{n} x_i \leq c$$

$$x_i \leq D_i \quad \forall i$$

$$x_i \geq 0 \quad \forall i$$

The solution to this is given by a simple greedy policy

$$V_n^{UB}(c) = \mathbb{E}\left[\sum_{k=1}^{n} p_k \cdot \min\left\{D_k, c - \sum_{i=1}^{k-1} D_i\right\}\right]$$

we can take expectation over $\{D_1, D_2, \ldots, D_n\}$

In words - allocate to all class 1 customers, then to class 2, and so on till seats are exhausted.

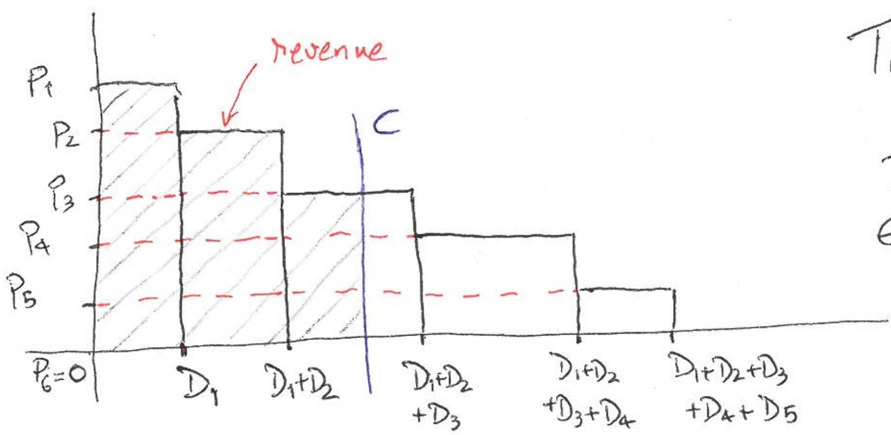
Now we try and simplify this

$$V_n^{UB}(c) = \mathbb{E}\left[\sum_{k=1}^{n} p_k \cdot \min\left\{D_k, c - \sum_{i=1}^{k-1} D_i\right\}\right]$$

$$= \mathbb{E}\Big[p_1 \cdot \min\{D_1, c\} + p_2 \cdot \min\{D_2, c - D_1\} + \dots + p_n \cdot \min\left\{D_n, c - \sum_{i=1}^{n-1} D_i\right\}\Big]$$

we define $p_{n+1} = 0$

$$= \mathbb{E}\left[\sum_{k=1}^{n} (p_k - p_{k+1}) \left[\min\left\{\sum_{i=1}^{k} D_i, c\right\}\right]\right]$$



The last equality follows from the diagram - essentially, it counts the revenue in two ways!

By linearity of expectation (we can always use that!!)

$$V_n^{UB}(c) = \sum_{k=1}^{n} (p_k - p_{k+1})\, \underbrace{\mathbb{E}\left[\min\left\{\sum_{i=1}^{k} D_i, c\right\}\right]}$$

By Jensen's: $\mathbb{E}[\min(\,)] \leq \min(\mathbb{E}[\,])$

$$\leq \sum_{k=1}^{n} (p_k - p_{k+1}) \cdot \min\left\{\sum_{i=1}^{k} \mu_i, c\right\}$$

$\mu_i = \mathbb{E}[D_i]$

So now we have 2 upper bounds

LP bound - $V_n^{LP}(c) = \mathbb{E}\left[\sum_{k=1}^{n} P_k \cdot \min\left\{D_k, c - \sum_{i=1}^{k-1} D_i\right\}\right]$

'Fluid' bound - $V_n^{Fl}(c) = \sum_{k=1}^{n} (P_k - P_{k+1}) \cdot \min\left\{\sum_{i=1}^{k} \mu_i, c\right\}$

Let $V_n(c)$ denote the actual value function. Then

$$\boxed{V_n(c) \leq V_n^{LP}(c) \leq V_n^{Fl}(c)}$$

From fluid LP to bid-prices

- Whenever you see an LP, always ask what the dual can tell us! Let's try this for the fluid LP

$$V_n^{Fl}(c) \equiv \max \sum_{k=1}^{n} P_k x_k$$

s.t.

$$x_k \leq \mu_k \ \forall k \quad : y_k$$

$$\sum_{k=1}^{n} x_k \leq c \quad : z$$

$$x_k \geq 0$$

$$V_n^{dual}(c) \equiv \min \sum_{k=1}^{n} \mu_k y_k + c \cdot z$$

s.t.

$$y_k + z \geq P_k \ \forall k$$

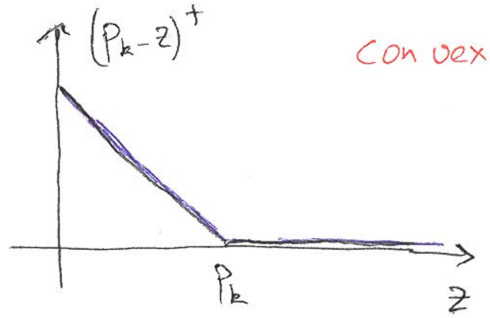
$$y_k \geq 0, \ z \geq 0$$

- Soln to dual: $y_k = (P_k - z)^+$, $V_n^{dual}(c) = \min_{z \geq 0} \left[\sum_{k=1}^{n} \mu_k (P_k - z)^+ + cz\right]$

$$\Rightarrow z(c) = \arg\min_{z \geq 0} \left[\underbrace{\sum_{k=1}^{n} \mu_k (P_k - z)^+ + cz}_{\text{convex!}}\right]$$

- Continuing from above

$$z(c) = \arg\min_{z \geq 0} \left[\sum_{k=1}^{n} \mu_k (P_k - z)^+ + c \cdot z \right]$$



convex!

For any $\mu_k, c \geq 0$

$\sum_{k=1}^{n} \mu_k (P_k - z)^+ + c \cdot z$ is convex

Let $z^* = \arg\min \left[\sum_{k=1}^{n} \mu_k (P_k - z)^+ + cz \right]$

$$\Rightarrow \boxed{\begin{aligned} z(c) &= \max \{ z^*, 0 \} \\ y_k &= (P_k - z(c))^+ \end{aligned}}$$

We can simplify this to get $z(c) = 0$ if $c > \sum_{k=1}^{n} \mu_k$

else $z(c) = \min \left\{ P_j \,\middle|\, c \leq \sum_{k=1}^{j} \mu_k \right\}$

This gives us a <u>bid-price heuristic policy</u>

- If $c \leq \sum_{k=1}^{j} \mu_k$: accept all customers
- Else, accept fare classes $A(c) = \{ j \mid \underbrace{P_j > z(c)}_{\text{i.e. } j \mid c \leq \sum_{k=1}^{j} \mu_k} \}$ bid-price