LP-based approx for Network RM

- **Setting**
  - m resources, n products, T periods
  - Resource i has initial capacity c_i
  - Product $j \equiv$ requirement vector A_j, price p_j

    $A_j = (a_{1j}, a_{2j}, \ldots, a_{mj})^T$, where a_{ij} is the amount of resource i required by prod j.

- **Bellman Eqn and Optimal Policy**
  - $\underline{x} = (x_1, x_2, \ldots, x_m)^T$: state vector
  - In period t, at most one request arrives
    - request is for product j with prob $\lambda_j(t)$
    - no request with prob $\lambda_0(t) = 1 - \sum_{j=1}^{n} \lambda_j(t)$
  - $\underline{u}(t, \underline{x}) = (u_1(t,\underline{x}), \underbrace{u_2(t,\underline{x})}, \ldots, u_n(t,\underline{x}))^T$

$$u_2(t,\underline{x}) = \begin{cases} 1 \ ; & \text{accept request for prod 2 at pd } t, \text{ state } \underline{x} \\ 0 \ ; & \text{reject request for prod 2 at pd } t, \text{ state } \underline{x} \end{cases}$$

– Bellman Eqn

$$V_t(\underline{x}) = \sum_{j=1}^{n} \lambda_j(t) \cdot \max_{\underbrace{u_j \in \{0,1\},\ A_j u_j \leq \underline{x}}_{\text{check for feasibility}}} \left[P_j u_j + V_{t+1}(\underline{x} - A_j u_j) \right] + \lambda_0(t) V_{t+1}(\underline{x})$$

$$V_{T+1}(\underline{x}) = 0 \quad \forall \ \underline{x}$$

(Alternately, can define $V_t(\underline{x}) = -\infty$ for all $\underline{x}$ s.t $x_i < 0$ for some i ≡ this also captures the feasibility check)

– Optimal Policy.

$$u_j^*(t, \underline{x}) = \begin{cases} 1 & \text{if } P_j \geq V_{t+1}(\underline{x} - A_j) - V_{t+1}(\underline{x}) \\ 0 & \text{if } P_j < \underbrace{V_{t+1}(\underline{x} - A_j) - V_{t+1}(\underline{x})}_{\text{marginal loss in value}} \end{cases}$$

– $\boxed{\text{Curse of Dimensionality}}$

To store $V_{t+1}(\underline{x})$, need $\prod_{i=1}^{m} (c_i + 1) \approx (c+1)^m$ states!

- ## LP-based Approximation

As for single-~~item~~ resource alloc^n, suppose all demand available simultaneously

- $$D_j = \sum_{t=1}^{T} \mathbb{1}\{\text{Request for } j \text{ arrived in period } t\}$$

$$= \sum_{t=1}^{T} X_{j,t} \quad , \quad \text{where } X_{j,t} \equiv \text{Bernoulli}(\lambda_j(t))$$

$$\mathbb{E}[D_j] \triangleq \mu_j = \sum_{t=1}^{T} \lambda_j(t)$$ (Linearity of expectation!)

- Now we have the following randomized-LP upper bound

$$V_T^{UB}(\underline{c}) \equiv \max \sum_{j=1}^{n} p_j \cdot y_j$$

$$\text{s.t} \quad \sum_{j=1}^{n} a_{ij}\, y_j \leq c_i \quad \forall i$$

$$0 \leq y_j \leq D_j \quad \forall j$$

Here $y_j \equiv$ # of requests for product j which we accept (i.e.: booking limit!)

- More generally, for any $(t, \underline{x})$, we have
  - $D_j[t,T] \triangleq \sum_{t'=t}^{T} X_{j,t'}$
  - $\mu_j[t,T] \triangleq \sum_{t'=t}^{T} \lambda_j(t')$
  - $V_t^{UB}(\underline{x}) \triangleq \max \sum_{j=1}^{n} p_j y_j$

    (given $D_j[t,T]$)

    Randomized LP

    $$\text{s.t. } \sum_{j=1}^{n} a_{ij} y_j \leq x_i \ \forall i$$

    $$0 \leq y_j \leq D_j[t,T] \ \forall j$$

  - $V_t^{FL}(\underline{x}) \triangleq \max \sum_{j=1}^{n} p_j y_j$

    Fluid LP

    $$\text{s.t. } \sum_{j=1}^{n} a_{ij} y_j \leq x_i \ \forall i$$

    $$0 \leq y_j \leq \mu_j[t,T] \ \forall j$$

  - Using our concavity arguments, we have

$$\boxed{V_t(\underline{x}) \leq \mathbb{E}\left[V_t^{UB}(\underline{x})\right] \leq V_t^{FL}(\underline{x})}$$

- Thus we have 2 ways to approximate $V_t(\underline{x})$ using linear programs.

- Now, from before, we have

$$u_j(t, \underline{x}) = \mathbb{1}\{ p_j > \underbrace{V_{t+1}(\underline{x}) - V_{t+1}(\underline{x} - A_j)}_{\text{denote by } \Delta_j V_{t+1}(\underline{x})} \}$$

we can now substitute the LP-based bounds to compute this!

- Problem: for each j, need to solve an LP (if using fluid UB, else several LPs) to get $V^{FL}_{t+1}(\underline{x} - A_j)$
- We will instead use the dual LP to get a bid-price policy.

- Consider the fluid LP $V^{FL}_T(\underline{c})$

(Primal)

$$\max \sum_{j=1}^{n} y_j p_j$$

$$\text{s.t} \quad \sum_{j=1}^{n} a_{ij} y_j \leq c_i \ \forall i \ : z_i$$

$$y_j \leq \mu_j \ \forall j \ : \beta_j$$

$$y_j \geq 0$$

(Dual)

$$\min \sum_{j=1}^{n} \mu_j \beta_j + \sum_{i=1}^{m} c_i z_i$$

s.t

$$\beta_j + \sum_{i=1}^{m} a_{ij} z_i \geq p_j \ \forall j$$

$$\beta_j \geq 0, \ z_i \geq 0$$

- Observe that given $\{z_i\}$, the optimal β_j are given by $\beta_j = \left(P_j - \sum_{i=1}^{m} a_{ij} z_i\right)^+$

$$\Rightarrow V_T^{FL}(\underline{c}) \equiv \min_{z_i \geq 0} \left\{ \underline{c}^T \underline{z} + \sum_{j=1}^{n} \mu_j \left(P_j - A_j^T z\right)^+ \right\}$$

- Suppose $\{\underline{y_j^*}\}$ are the primal solution $\{\underline{z}^*, \underline{\beta}^*\}$ are the dual solution

Then by complementary slackness

$$z_i^* > 0 \Rightarrow \sum_{j=1}^{n} a_{ij} y_j = c_i$$

$$\beta_j^* > 0 \Rightarrow y_j = \mu_j$$

One way to interpret this is that we pay a cost of z_i^* per unit of resource i (and similarly, a cost of β_j^* per additional customer requesting product j).

Bid-prices

- The dual solution also suggests the following bid-price policy - For given state $(t, \underline{x})$
  - Compute $\{z_i^*\}$ from the dual $V_t^{FL}(\underline{x})$
  - For each product j, associate a bid-price $\sum_{i=1}^{m} a_{ij} z_i^*$
  - Accept product j iff $p_j \geq \sum_{i=1}^{m} a_{ij} z_i^*$ (else reject)

- One way to interpret this is that we are approximating $V_{t+1}(\underline{x}) \approx \sum_{i=1}^{m} x_i z_i^*$

  $\Rightarrow \quad \Delta_j V_{t+1}(\underline{x}) = V_{t+1}(\underline{x}) - V_{t+1}(\underline{x} - A_j) = \sum_{i=1}^{m} a_{ij} z_i^*$

- To get a static policy, we compute bid prices using $V_T^{FL}(\underline{c})$. To get dynamic policy, we can update bid-prices using $V_t^{FL}(\underline{x})$ for $(t, \underline{x})$

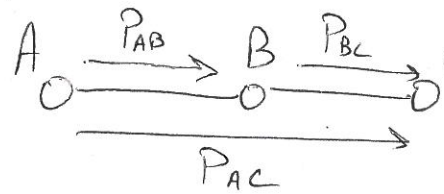
Eg - For a hotel, we can compute a bid-price for each night $(Z_M, Z_{Tu}, \dots, Z_{Sun})$.

Now if a customer wants to book (Tu, W, Th) for price P, we accept only if

$$P \geq Z_{Tu} + Z_W + Z_{Th}$$

Eg - (Bid prices are not always optimal)

This method of obtaining bid-prices is more general than our LP argument; given any approx $V_t(\underline{x}) \approx \sum_{i=1}^{m} x_i W_i$, we can set $\{W_i\}$ as the bid-prices. However, this may not always be optimal.

- Consider network  , $C_{AB} = 1$, $C_{BC} = 1$

- Suppose $P_{AB} = P_{BC} = 250$, $P_{AC} = 450$

- Consider 2 periods, with $\lambda(1) = (\overset{\lambda_{AB}}{0.3}, \overset{\lambda_{BC}}{0.3}, \overset{\lambda_{AC}}{0.4})$

$$\lambda(2) = (0, 0, 0.8)$$

- Claim 1

The optimal policy is to only accept request for AC in each period.

To see this, notice resulting revenue $= 450.(0.4 + 0.6 \times 0.8)$
$= 396$

On the other hand, accepting AB or BC at $t=1$ gives at most 250 (as we can not accept AC at $t=2$)

- Claim 2 - This can not be implemented by bid prices.

To see this, note that bid-prices (Z_{AB}, Z_{BC}) must satisfy

$$Z_{AB} > 250, \quad Z_{BC} > 250$$
$$Z_{AB} + Z_{BC} \leq 450$$

This is not possible!

However, bid-prices are good whenever c, T are large (see HW 2!)