ORIE 4154 - Pricing and Market Design

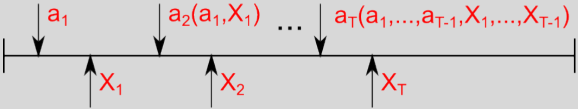
Module 1: Capacity-based Revenue Management (Multiple Fare-Class Capacity Allocation)

Instructor: Sid Banerjee, ORIE

Cornell University

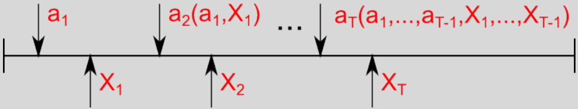
(Stochastic) Dynamic Programming

Sequential decision making: $\max_{a:\text{"Actions"}} \mathbb{E}_X[f(a,X)]$



(Stochastic) Dynamic Programming

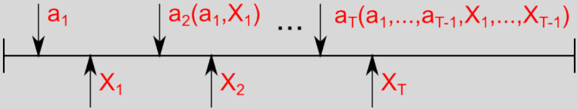
Sequential decision making: $\max_{a:\text{“Actions”}} \mathbb{E}_X[f(a,X)]$



5 Components of formulation

(Stochastic) Dynamic Programming

Sequential decision making: $\max_{a:\text{“Actions”}} \mathbb{E}_X[f(a,X)]$

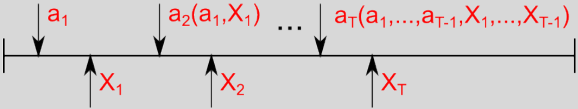


5 Components of formulation

- Horizon: T - discrete ‘decision periods’ $t = \{1, 2, \ldots, T\}$

(Stochastic) Dynamic Programming

Sequential decision making: $\max_{a:\text{"Actions"}} \mathbb{E}_X[f(a,X)]$

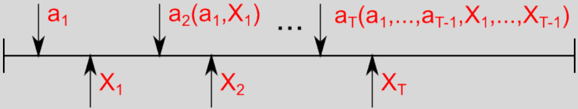


5 Components of formulation

- Horizon: T - discrete 'decision periods' $t = \{1, 2, \ldots, T\}$
- State: $s_t \in \mathscr{S}_t$ - concise summary of history

(Stochastic) Dynamic Programming

Sequential decision making: $\max_{a:\text{“Actions”}} \mathbb{E}_X[f(a,X)]$

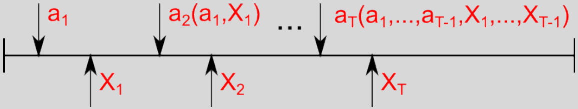


5 Components of formulation

- Horizon: T - discrete ‘decision periods’ $t = \{1, 2, \ldots, T\}$
- State: $s_t \in \mathscr{S}_t$ - concise summary of history
- Action: $a_t \in \mathscr{A}(s_t)$ - allowed set actions in each period

(Stochastic) Dynamic Programming

Sequential decision making: $\max_{a:\text{“Actions”}} \mathbb{E}_X[f(a,X)]$

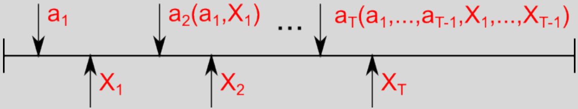


5 Components of formulation

- Horizon: T - discrete ‘decision periods’ $t = \{1,2,\dots,T\}$
- State: $s_t \in \mathscr{S}_t$ - concise summary of history
- Action: $a_t \in \mathscr{A}(s_t)$ - allowed set actions in each period
- Randomness/Disturbance: X_t - determines state transition probability $p(s_{t+1}|s_t,a_t)$ (or $s_{t+1} = f(s_t,a_t,X_t)$)

(Stochastic) Dynamic Programming

Sequential decision making: $\max_{a:\text{"Actions"}} \mathbb{E}_X[f(a,X)]$

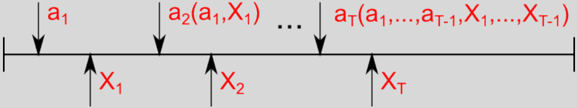


5 Components of formulation

- Horizon: T - discrete 'decision periods' $t = \{1, 2, \ldots, T\}$
- State: $s_t \in \mathscr{S}_t$ - concise summary of history
- Action: $a_t \in \mathscr{A}(s_t)$ - allowed set actions in each period
- Randomness/Disturbance: X_t - determines state transition probability $p(s_{t+1}|s_t, a_t)$ (or $s_{t+1} = f(s_t, a_t, X_t)$)
- Reward: $R_t(s_t, a_t, X_t)$ (or $R_t(s_{t+1}|s_t, a_t)$)

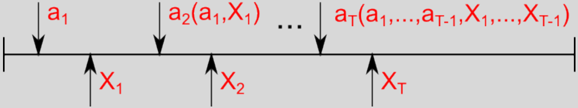
Backward induction

Sequential decision making: $\max_{a:\text{“Actions”}} \mathbb{E}_X[f(a,X)]$



Backward induction

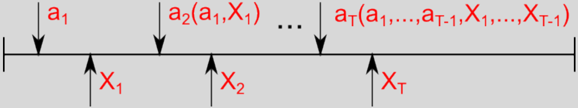
Sequential decision making: $\max_{a:\text{"Actions"}} \mathbb{E}_X[f(a,X)]$



5 Components of formulation

Backward induction

Sequential decision making: $\max_{a:\text{"Actions"}} \mathbb{E}_X[f(a,X)]$

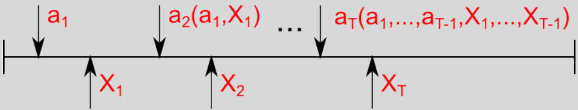


5 Components of formulation

- Horizon: T - discrete 'decision periods' $t = \{1, 2, \ldots, T\}$

Backward induction

Sequential decision making: $\max_{a:\text{"Actions"}} \mathbb{E}_X[f(a,X)]$

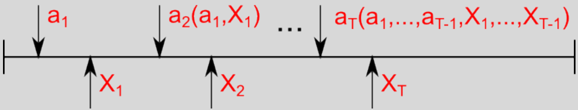


5 Components of formulation

- Horizon: T - discrete ‘decision periods’ $t = \{1, 2, \ldots, T\}$
- State: $s_t \in \mathscr{S}_t$ - concise summary of history

Backward induction

Sequential decision making: $\max_{a:\text{"Actions"}} \mathbb{E}_X[f(a,X)]$

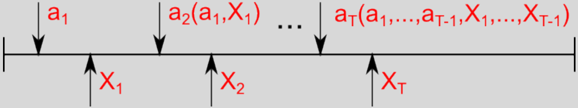


5 Components of formulation

- Horizon: T - discrete 'decision periods' $t = \{1, 2, \ldots, T\}$
- State: $s_t \in \mathscr{S}_t$ - concise summary of history
- Action: $a_t \in \mathscr{A}(s_t)$ - allowed set actions in each period

Backward induction

Sequential decision making: $\max_{a:\text{"Actions"}} \mathbb{E}_X[f(a,X)]$

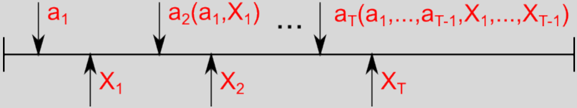


5 Components of formulation

- Horizon: T - discrete 'decision periods' $t = \{1,2,\ldots,T\}$
- State: $s_t \in \mathscr{S}_t$ - concise summary of history
- Action: $a_t \in \mathscr{A}(s_t)$ - allowed set actions in each period
- Randomness/Disturbance: X_t - determines state transition probability $p(s_{t+1}|s_t,a_t)$ (or $S_{t+1}(s_t,a_t,X_t)$)

Backward induction

Sequential decision making: $\max_{a:\text{"Actions"}} \mathbb{E}_X[f(a,X)]$

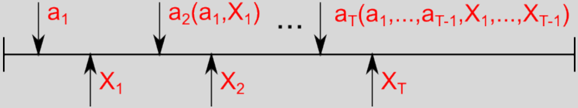


5 Components of formulation

- Horizon: T - discrete 'decision periods' $t = \{1,2,\ldots,T\}$
- State: $s_t \in \mathscr{S}_t$ - concise summary of history
- Action: $a_t \in \mathscr{A}(s_t)$ - allowed set actions in each period
- Randomness/Disturbance: X_t - determines state transition probability $p(s_{t+1}|s_t,a_t)$ (or $S_{t+1}(s_t,a_t,X_t)$)
- Reward: $R_t(s_t,a_t,X_t)$ (or $R_t(s_{t+1}|s_t,a_t)$)

Backward induction

Sequential decision making: $\max_{a:\text{"Actions"}} \mathbb{E}_X[f(a,X)]$



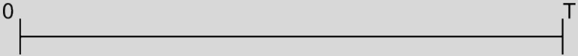
5 Components of formulation

- Horizon: T - discrete ‘decision periods’ $t = \{1, 2, \ldots, T\}$
- State: $s_t \in \mathscr{S}_t$ - concise summary of history
- Action: $a_t \in \mathscr{A}(s_t)$ - allowed set actions in each period
- Randomness/Disturbance: X_t - determines state transition probability $p(s_{t+1}|s_t, a_t)$ (or $S_{t+1}(s_t, a_t, X_t)$)
- Reward: $R_t(s_t, a_t, X_t)$ (or $R_t(s_{t+1}|s_t, a_t)$)

Reformulated Problem: $\max_{a_1(S_1), a_2(S_2), \ldots, a_T(S_T)} \sum_{t=1}^{T} \mathbb{E}[R_t(S_t, a_t, X_t)]$

Backward Induction

- Horizon: T
- State: $s_t \in \mathscr{S}_t$
- Action: $a_t \in \mathscr{A}(s_t)$
- Randomness: X_t - determines $p(s_{t+1}|s_t, a_t)$ (or $S_{t+1}(s_t, a_t, X_t)$)
- Reward: $R_t(s_t, a_t, X_t)$ (or $R_t(s_{t+1}|s_t, a_t)$)

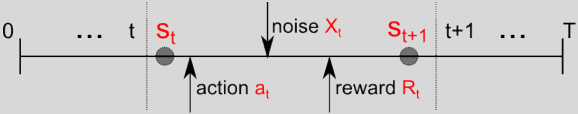


Backward Induction

- Horizon: T • State: $s_t \in \mathscr{S}_t$ • Action: $a_t \in \mathscr{A}(s_t)$
- Randomness: X_t - determines $p(s_{t+1}|s_t, a_t)$ (or $S_{t+1}(s_t, a_t, X_t)$)
- Reward: $R_t(s_t, a_t, X_t)$ (or $R_t(s_{t+1}|s_t, a_t)$)

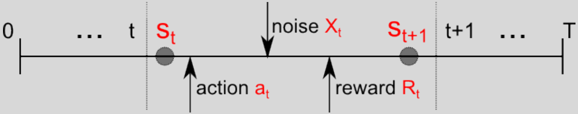
Backward Induction

- Horizon: T • State: $s_t \in \mathscr{S}_t$ • Action: $a_t \in \mathscr{A}(s_t)$
- Randomness: X_t - determines $p(s_{t+1}|s_t, a_t)$ (or $S_{t+1}(s_t, a_t, X_t)$)
- Reward: $R_t(s_t, a_t, X_t)$ (or $R_t(s_{t+1}|s_t, a_t)$)



Backward Induction

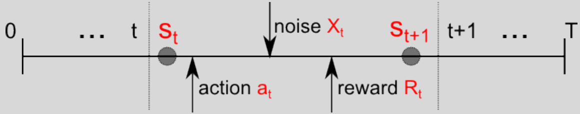
- Horizon: T • State: $s_t \in \mathscr{S}_t$ • Action: $a_t \in \mathscr{A}(s_t)$
- Randomness: X_t - determines $p(s_{t+1}|s_t,a_t)$ (or $S_{t+1}(s_t,a_t,X_t)$)
- Reward: $R_t(s_t,a_t,X_t)$ (or $R_t(s_{t+1}|s_t,a_t)$)



- Value function: $V_t(s) \triangleq$ Maximum possible total expected reward over periods $\{t, t+1, \ldots, T\}$

Backward Induction

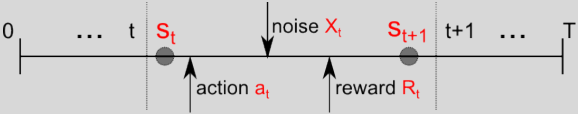
- Horizon: T • State: $s_t \in \mathscr{S}_t$ • Action: $a_t \in \mathscr{A}(s_t)$
- Randomness: X_t - determines $p(s_{t+1}|s_t, a_t)$ (or $S_{t+1}(s_t, a_t, X_t)$)
- Reward: $R_t(s_t, a_t, X_t)$ (or $R_t(s_{t+1}|s_t, a_t)$)



- Value function: $V_t(s) \triangleq$ Maximum possible total expected reward over periods $\{t, t+1, \ldots, T\}$
- Terminal reward: $V_T(s) = \max\limits_{a \in \mathscr{A}(s)} \sum_{x \in \mathscr{S}} p(x|s,a) R_T(x|s,a)$

Backward Induction

- Horizon: T • State: $s_t \in \mathscr{S}_t$ • Action: $a_t \in \mathscr{A}(s_t)$
- Randomness: X_t - determines $p(s_{t+1}|s_t,a_t)$ (or $S_{t+1}(s_t,a_t,X_t)$)
- Reward: $R_t(s_t,a_t,X_t)$ (or $R_t(s_{t+1}|s_t,a_t)$)



- Value function: $V_t(s) \triangleq$ Maximum possible total expected reward over periods $\{t, t+1, \ldots, T\}$
- Terminal reward: $V_T(s) = \max_{a\in\mathscr{A}(s)} \sum_{x\in\mathscr{S}} p(x|s,a) R_T(x|s,a)$
- Bellman Optimality Equation:

$$\begin{aligned} V_t(s) &= \max_{a\in\mathscr{A}(s)} \mathbb{E}\left[R_t(s,a,X_t) + V_{t+1}(S_{t+1}(s,a,X_t))\right] \\ &= \max_{a\in\mathscr{A}(s)} \sum_x p(x|s,a)\left[R_t(x|s,a) + V_{t+1}(x)\right] \end{aligned}$$

Dynamic Programming: Example

- Horizon: T • State: $s_t \in \mathscr{S}_t$ • Action: $a_t \in \mathscr{A}(s_t)$
- Randomness: X_t - determines $p(s_{t+1}|s_t, a_t)$ (or $S_{t+1}(s_t, a_t, X_t)$)
- Reward: $R_t(s_t, a_t, X_t)$ (or $R_t(s_{t+1}|s_t, a_t)$)
- Value function: $V_t(s) \triangleq$ Max $\mathbb{E}[R_t + R_{t+1} + \ldots + R_T]$
- Bellman Eqn: $V_t(s) = \max_{a \in \mathscr{A}(s)} \sum_x p(x|s,a) \left[R_t(x|s,a) + V_{t+1}(x)\right]$

Dynamic Programming: Example

- Horizon: T
- State: $s_t \in \mathscr{S}_t$
- Action: $a_t \in \mathscr{A}(s_t)$
- Randomness: X_t - determines $p(s_{t+1}|s_t, a_t)$ (or $S_{t+1}(s_t, a_t, X_t)$)
- Reward: $R_t(s_t, a_t, X_t)$ (or $R_t(s_{t+1}|s_t, a_t)$)
- Value function: $V_t(s) \triangleq$ Max $\mathbb{E}[R_t + R_{t+1} + \ldots + R_T]$
- Bellman Eqn: $V_t(s) = \max_{a \in \mathscr{A}(s)} \sum_x p(x|s,a) \left[R_t(x|s,a) + V_{t+1}(x)\right]$

Toothpick game (from last class)

- State: $s_t = \#$ of toothpicks at beginning of epoch t
- Actions: Pick $\mathscr{A}(s_t) = \{1, 2\}$ toothpicks
- Randomness: $X_t \sim$UNIF$\{1, 2\}$; $S_{t+1}(s_t, a_t, X_t) = s_t - a_t - X_t$
- Reward: $R_t(s_t, a_t, X_t) = \mathbb{1}_{\{s_t > 0, s_t - a_t = 0\}}$
- Bellman Eqn:

$$V_t(s) = \max_{a \in \{1,2\}} \left[0.5 \cdot [V_{t+1}(s-a-1) + V_{t+1}(s-a-2)]\right]$$

Single-resource multi-stage capacity allocation

- Seller constraints:
  - (Setting) Capacity C
    n fare-classes with prices $p_1 > p_2 > p_3 > \ldots > p_n$

Single-resource multi-stage capacity allocation

- Seller constraints:
  - (Setting) Capacity C
    n fare-classes with prices $p_1 > p_2 > p_3 > \ldots > p_n$
- Buyer behavior and information structure:
  - (Perfect Segmentation) Fare-classes arrive separately
  - (Dynamics) Sequential arrival in increasing order of fares
  - (Demand) Independent demand for fare-class i: $D_i \sim F_i$

Single-resource multi-stage capacity allocation

- Seller constraints:
  - (Setting) Capacity C
    n fare-classes with prices $p_1 > p_2 > p_3 > \ldots > p_n$
- Buyer behavior and information structure:
  - (Perfect Segmentation) Fare-classes arrive separately
  - (Dynamics) Sequential arrival in increasing order of fares
  - (Demand) Independent demand for fare-class i: $D_i \sim F_i$

DP Formulation

- n periods: $\{n, n-1, \ldots, 1\}$

Single-resource multi-stage capacity allocation

- Seller constraints:
  - (Setting) Capacity C
    n fare-classes with prices $p_1 > p_2 > p_3 > \ldots > p_n$
- Buyer behavior and information structure:
  - (Perfect Segmentation) Fare-classes arrive separately
  - (Dynamics) Sequential arrival in increasing order of fares
  - (Demand) Independent demand for fare-class i: $D_i \sim F_i$

DP Formulation

- n periods: $\{n, n-1, \ldots, 1\}$
- State: $s_k =$ available capacity when fare-class k arrives

Single-resource multi-stage capacity allocation

- Seller constraints:
  - (Setting) Capacity C
    n fare-classes with prices $p_1 > p_2 > p_3 > \ldots > p_n$
- Buyer behavior and information structure:
  - (Perfect Segmentation) Fare-classes arrive separately
  - (Dynamics) Sequential arrival in increasing order of fares
  - (Demand) Independent demand for fare-class i: $D_i \sim F_i$

DP Formulation

- n periods: $\{n, n-1, \ldots, 1\}$
- State: $s_k =$ available capacity when fare-class k arrives
- Action: Allocate $s_k - y_k$ capacity at fare p_k

Single-resource multi-stage capacity allocation

- Seller constraints:
  - (Setting) Capacity C
    n fare-classes with prices $p_1 > p_2 > p_3 > \ldots > p_n$
- Buyer behavior and information structure:
  - (Perfect Segmentation) Fare-classes arrive separately
  - (Dynamics) Sequential arrival in increasing order of fares
  - (Demand) Independent demand for fare-class i: $D_i \sim F_i$

DP Formulation

- n periods: $\{n, n-1, \ldots, 1\}$
- State: $s_k =$ available capacity when fare-class k arrives
- Action: Allocate $s_k - y_k$ capacity at fare p_k
- Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$

$$V_k(s_k) = \max_{y_k \in \{0,\ldots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\left(\max\{s_k - D_k, y_k\}\right)\right]$$

Single-resource multi-stage capacity allocation

DP Formulation

- State: s_k = available capacity when fare-class k arrives
- Action: Allocate $s_k - y_k$ capacity at fare p_k
- Bellman Eqn: For $k \in \{n, n-1, \dots, 2\}$

$$V_k(s_k) = \max_{y_k \in \{0,\dots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\left(\max\{s_k - D_k, y_k\}\right)\right]$$

Single-resource multi-stage capacity allocation

DP Formulation

- State: $s_k =$ available capacity when fare-class k arrives
- Action: Allocate $s_k - y_k$ capacity at fare p_k
- Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$
  $V_k(s_k) = \max_{y_k \in \{0,\ldots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\big(\max\{s_k - D_k, y_k\}\big)\right]$

To solve this: use backward induction

Single-resource multi-stage capacity allocation

DP Formulation

- State: $s_k =$ available capacity when fare-class k arrives
- Action: Allocate $s_k - y_k$ capacity at fare p_k
- Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$
  $$V_k(s_k) = \max_{y_k \in \{0,\ldots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\big(\max\{s_k - D_k, y_k\}\big)\right]$$

To solve this: use backward induction

- For $s_1 \in \{0, 1, \ldots, C\}$, we have $V_1(s_1) = \mathbb{E}[p_1 \max\{s_1, D_1\}]$

Single-resource multi-stage capacity allocation

DP Formulation

- State: $s_k =$ available capacity when fare-class k arrives
- Action: Allocate $s_k - y_k$ capacity at fare p_k
- Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$
$$V_k(s_k) = \max_{y_k \in \{0,\ldots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\left(\max\{s_k - D_k, y_k\}\right)\right]$$

To solve this: use backward induction

- For $s_1 \in \{0, 1, \ldots, C\}$, we have $V_1(s_1) = \mathbb{E}[p_1 \max\{s_1, D_1\}]$
- Given $V_{k-1}(\cdot)$, can compute $V_k(\cdot)$ via Bellman Eqn.

Single-resource multi-stage capacity allocation

DP Formulation

- State: $s_k =$ available capacity when fare-class k arrives
- Action: Allocate $s_k - y_k$ capacity at fare p_k
- Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$
  $V_k(s_k) = \max_{y_k \in \{0,\ldots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\left(\max\{s_k - D_k, y_k\}\right)\right]$

To solve this: use backward induction

- For $s_1 \in \{0, 1, \ldots, C\}$, we have $V_1(s_1) = \mathbb{E}[p_1 \max\{s_1, D_1\}]$
- Given $V_{k-1}(\cdot)$, can compute $V_k(\cdot)$ via Bellman Eqn.
- If D_is are discrete rv on $\{0, 1, \ldots, C\}$, then running time is nC^2
  (# of periods $\times$# of states $\times$ time to compute expectation)

Single-resource multi-stage capacity allocation

DP Formulation

- State: $s_k =$ available capacity when fare-class k arrives
- Action: Allocate $s_k - y_k$ capacity at fare p_k
- Bellman Eqn: For $k \in \{n, n-1, \dots, 2\}$
  $$V_k(s_k) = \max_{y_k \in \{0,\dots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\left(\max\{s_k - D_k, y_k\}\right)\right]$$

To solve this: use backward induction

- For $s_1 \in \{0, 1, \dots, C\}$, we have $V_1(s_1) = \mathbb{E}[p_1 \max\{s_1, D_1\}]$
- Given $V_{k-1}(\cdot)$, can compute $V_k(\cdot)$ via Bellman Eqn.
- If D_is are discrete rv on $\{0, 1, \dots, C\}$, then running time is nC^2
  (# of periods $\times$# of states $\times$ time to compute expectation)

Unsatisfying - not interpretable, highly model dependent

Single-resource multi-stage capacity allocation

DP Formulation

- State: $s_k =$ available capacity when fare-class k arrives
- Action: Allocate $s_k - y_k$ capacity at fare p_k
- Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$
  $$V_k(s_k) = \max_{y_k \in \{0,\ldots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\left(\max\{s_k - D_k, y_k\}\right)\right]$$

To solve this: use backward induction

- For $s_1 \in \{0, 1, \ldots, C\}$, we have $V_1(s_1) = \mathbb{E}[p_1 \max\{s_1, D_1\}]$
- Given $V_{k-1}(\cdot)$, can compute $V_k(\cdot)$ via Bellman Eqn.
- If D_is are discrete rv on $\{0, 1, \ldots, C\}$, then running time is nC^2 (# of periods $\times$# of states $\times$ time to compute expectation)

Unsatisfying - not interpretable, highly model dependent
Would like to understand structure of optimal solution

Single-resource multi-stage capacity allocation

- $s_k =$ available capacity for fare-class k; allocate $s_k - y_k$ at p_k
- Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$
  $V_k(s_k) = \max_{y_k \in \{0,\ldots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\left(\max\{s_k - D_k, y_k\}\right)\right]$

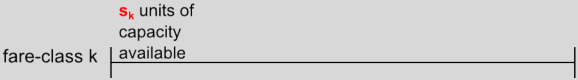
Suppose instead we know D_K before choosing y_k

Single-resource multi-stage capacity allocation

- s_k = available capacity for fare-class k; allocate $s_k - y_k$ at p_k
- Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$

$$V_k(s_k) = \max_{y_k \in \{0,\ldots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\left(\max\{s_k - D_k, y_k\}\right)\right]$$

Suppose instead we know D_K before choosing y_k

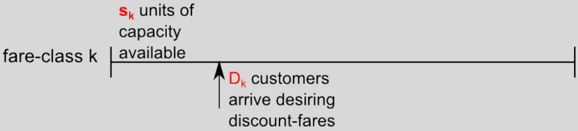


Single-resource multi-stage capacity allocation

- $s_k =$ available capacity for fare-class k; allocate $s_k - y_k$ at p_k
- Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$

$$V_k(s_k) = \max_{y_k \in \{0,\ldots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\left(\max\{s_k - D_k, y_k\}\right)\right]$$

Suppose instead we know D_K before choosing y_k

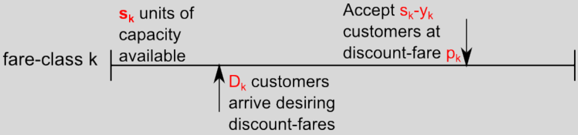


Single-resource multi-stage capacity allocation

- s_k = available capacity for fare-class k; allocate $s_k - y_k$ at p_k
- Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$

$$V_k(s_k) = \max_{y_k \in \{0,\ldots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\big(\max\{s_k - D_k, y_k\}\big)\right]$$

Suppose instead we know D_K before choosing y_k

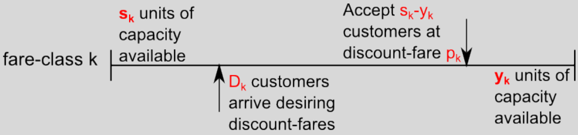


Single-resource multi-stage capacity allocation

- $s_k =$ available capacity for fare-class k; allocate $s_k - y_k$ at p_k
- Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$

$$V_k(s_k) = \max_{y_k \in \{0,\ldots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\big(\max\{s_k - D_k, y_k\}\big)\right]$$

Suppose instead we know D_K before choosing y_k

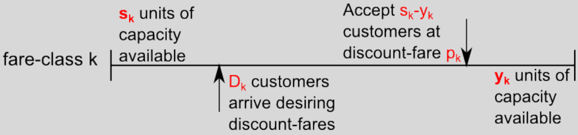


Single-resource multi-stage capacity allocation

- s_k = available capacity for fare-class k; allocate $s_k - y_k$ at p_k
- Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$

$$V_k(s_k) = \max_{y_k \in \{0,\ldots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\left(\max\{s_k - D_k, y_k\}\right)\right]$$

Suppose instead we know D_K before choosing y_k



- Let $\widehat{V}_k(s_k|D_k)$ = value fn given D_k ; $\widehat{V}_k(s_k) = \mathbb{E}[\widehat{V}_k(s_k|D_k)]$

$$\widehat{V}_k(s_k, D_k) = \max_{y_k \in \{(s_k - D_k)^+,\ldots,s_k\}} p_k \cdot (s_k - y_k) + \mathbb{E}\left[\widehat{V}_{k-1}(y_k|D_{k-1})\right]$$

- Modified Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$

$$\widehat{V}_k(s_k) = p_k s_k + \mathbb{E}\left[\max_{y_k \in \{(s_k - D_k)^+,\ldots,s_k\}} \left\{-p_k y_k + \widehat{V}_{k-1}(y_k)\right\}\right]$$

Single-resource multi-stage capacity allocation

- Original Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$
$$V_k(s_k) = \max_{y_k \in \{0,\ldots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\big(\max\{s_k - D_k, y_k\}\big)\right]$$

- $\widehat{V}_k(s_k|D_k)$ = value fn given D_k ; $\widehat{V}_k(s_k) = \mathbb{E}[\widehat{V}_k(s_k|D_k)]$
$$\widehat{V}_k(s_k, D_k) = \max_{y_k \in \{(s_k - D_k)^+,\ldots,s_k\}} p_k \cdot (s_k - y_k) + \mathbb{E}\left[\widehat{V}_{k-1}(y_k|D_{k-1})\right]$$
- Modified Bellman Eqn: For $k \in \{n, n-1, \ldots, 2\}$
$$\widehat{V}_k(s_k) = p_k s_k + \mathbb{E}\left[\max_{y_k \in \{(s_k - D_k)^+,\ldots,s_k\}} \left\{-p_k y_k + \widehat{V}_{k-1}(y_k)\right\}\right]$$

Single-resource multi-stage capacity allocation

- Original Bellman Eqn: For $k \in \{n, n-1, \dots, 2\}$
  $$V_k(s_k) = \max_{y_k \in \{0,\dots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\left(\max\{s_k - D_k, y_k\}\right)\right]$$

- $\widehat{V}_k(s_k|D_k) =$ value fn given D_k ; $\widehat{V}_k(s_k) = \mathbb{E}[\widehat{V}_k(s_k|D_k)]$
  $$\widehat{V}_k(s_k, D_k) = \max_{y_k \in \{(s_k - D_k)^+,\dots,s_k\}} p_k \cdot (s_k - y_k) + \mathbb{E}\left[\widehat{V}_{k-1}(y_k|D_{k-1})\right]$$
- Modified Bellman Eqn: For $k \in \{n, n-1, \dots, 2\}$
  $$\widehat{V}_k(s_k) = p_k s_k + \mathbb{E}\left[\max_{y_k \in \{(s_k - D_k)^+,\dots,s_k\}} \left\{-p_k y_k + \widehat{V}_{k-1}(y_k)\right\}\right]$$

Proposition: $\widehat{V}_k(s) \geq V_k(s)$ for all $k \in \{n, n-1, \dots, 1\}$, $s \in \mathbb{N}$

Single-resource multi-stage capacity allocation

- Original Bellman Eqn: For $k \in \{n, n-1, \dots, 2\}$
  $V_k(s_k) = \max_{y_k \in \{0,\dots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\big(\max\{s_k - D_k, y_k\}\big)\right]$

- $\widehat{V}_k(s_k|D_k) =$ value fn given D_k ; $\widehat{V}_k(s_k) = \mathbb{E}[\widehat{V}_k(s_k|D_k)]$
  $\widehat{V}_k(s_k, D_k) = \max_{y_k \in \{(s_k - D_k)^+,\dots,s_k\}} p_k \cdot (s_k - y_k) + \mathbb{E}\left[\widehat{V}_{k-1}(y_k|D_{k-1})\right]$
- Modified Bellman Eqn: For $k \in \{n, n-1, \dots, 2\}$
  $\widehat{V}_k(s_k) = p_k s_k + \mathbb{E}\left[\max_{y_k \in \{(s_k - D_k)^+,\dots,s_k\}} \left\{-p_k y_k + \widehat{V}_{k-1}(y_k)\right\}\right]$

Proposition: $\widehat{V}_k(s) \geq V_k(s)$ for all $k \in \{n, n-1, \dots, 1\}$, $s \in \mathbb{N}$

- Can always ignore D_k and choose fixed y_k

Single-resource multi-stage capacity allocation

- Original Bellman Eqn: For $k \in \{n, n-1, \dots, 2\}$
  $$V_k(s_k) = \max_{y_k \in \{0,\dots,s_k\}} \mathbb{E}\left[p_k \cdot \min\{D_k, s_k - y_k\} + V_{k-1}\left(\max\{s_k - D_k, y_k\}\right)\right]$$

- $\widehat{V}_k(s_k|D_k) =$ value fn given D_k ; $\widehat{V}_k(s_k) = \mathbb{E}[\widehat{V}_k(s_k|D_k)]$
  $$\widehat{V}_k(s_k, D_k) = \max_{y_k \in \{(s_k - D_k)^+,\dots,s_k\}} p_k \cdot (s_k - y_k) + \mathbb{E}\left[\widehat{V}_{k-1}(y_k|D_{k-1})\right]$$
- Modified Bellman Eqn: For $k \in \{n, n-1, \dots, 2\}$
  $$\widehat{V}_k(s_k) = p_k s_k + \mathbb{E}\left[\max_{y_k \in \{(s_k - D_k)^+,\dots,s_k\}} \left\{-p_k y_k + \widehat{V}_{k-1}(y_k)\right\}\right]$$

Proposition: $\widehat{V}_k(s) \geq V_k(s)$ for all $k \in \{n, n-1, \dots, 1\}$, $s \in \mathbb{N}$

- Can always ignore D_k and choose fixed y_k
- If we can design policy with $V_k(\cdot) = \widehat{V}_k(\cdot)$, then we are done!