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Problem 1: Choice models and assortment optimization

Part (a)

Consider a MNL choice model over five products with prices $(p1,\ldots,p5) = (7,6,4,3,2)$ and preference weights (i.e., MNL parameters) $(v1,\ldots,v5) = (3,5,6,4,5)$. The preference weight of the no-purchase alternative is $v_0 = 10$. Compute the optimal expected-revenue assortment.

Solution: Since the prices are decreasing, we consider all the nested options $\{1,\ldots,i\}$. Given such a set, the expected revenue is

$$\frac{\sum_{j=1}^{i} v_j p_j}{v_0 + \sum_{j=1}^{i} v_j}.$$

Computing these values, the best set is $\{1,2,3\}$.

Part (b)

Next, consider a mixed-MNL choice model, wherein we have two consumer types and three products. The probability of observing each consumer type is $(\alpha_1,\alpha_2) = (0.5,0.5)$. The product prices are $(p1,p2,p3) = (8,4,3)$. A consumer of type 1 has preference weights $(v11,v21,v31) = (5,20,0)$, and a consumer of type 2 has preference weights $(v12,v22,v32) = (1/5,10,10)$; the preference weight of the no-purchase alternative is 1 for both types.

i. First, find the optimal assortments $S_1^*,, S_2^*$ for each individual type, and compute the expected revenue of S_1^* and S_2^* for the mixed-MNL model.

   Solution: **(i)** Proceeding as in part **(a)**, we get that for type 1, the optimal assortment is $S_1^* = \{1\}$, and for type 2, the optimal assortment is $S_2^* = \{1,2\}$. Now we have:

$$\mathcal{R}(\{1\}) = \frac{1}{2}\left(\frac{5p_1}{1+5} + \frac{p_1/5}{1+1/5}\right) = 4$$
$$\mathcal{R}(\{1,2\}) = \frac{1}{2}\left(\frac{5p_1+20p_2}{1+5+20} + \frac{p_1/5+10p_2}{1+1/5+10}\right) \approx 4.16$$

ii. Next, consider the assortment $\{1,3\}$, and show that this achieves a higher revenue under the mixed-MNL model than the two assortments in the previous part. (In fact, $\{1,3\}$ is the optimal assortment, but you do not need to show that).

(ii) The assortment has revenue

$$\mathcal{R}(\{1,3\}) = \frac{1}{2}\left(\frac{5p_1}{1+5} + \frac{p_1/5+10p_3}{1+1/5+10}\right) \approx 4.74$$

Thus, this has a higher revenue than S_1^*, S_2^*.

Problem 2: Pivot rules and procurement auctions

Consider a single-item auction setting, where each bidder i has a private value v_i. In class, we studied the Vickrey (or second-price) auction for such settings, and saw that it has two properties: *i. incentive compatibility* (DSIC), i.e., $u_i(b_i, \mathbf{b}_{-i}) \leq u_i(v_i, \mathbf{b}_{-i})$, and *ii. individual rationality*, i.e., $u_i(v_i, \mathbf{b}_{-i}) \geq 0$. We now will see a more general mechanism that has the DSIC property.

Given bids $\mathbf{b}$, let $i^* = \arg\max_i\{b_i\}$, and consider the mechanism $(\mathbf{x}, \mathbf{p})$, with allocation rule $x_i = \mathbb{1}_{\{i=i^*\}}$ (i.e., award item to highest bidder), and payment rule $p_j = 0$ for all $j \neq i^*$, and $p_{i^*} = f_{i^*}(\mathbf{b}_{-i^*})$, where $f_i(\mathbf{b}_{-i^*})$ is some function which only depends on *the bids of other bidders* (and any other publicly-known constant). The term $f_{i^*}(\mathbf{b}_{-i^*})$ is sometimes referred to as a *pivot rule.*

Part (a)

Argue that $f_i(\mathbf{b}_{-i}) = \max_{j\neq i}\{b_j\}$ always ensures individual rationality in any single-item auction setting. This gives us the Vickrey auction (and more generally, the pivot rule is a special case of the so-called *Clark pivot rule*).

Solution: Assume that player i reports v_i. In the case his valuation is not the maximum his utility is zero. In the other case, he will get the item and pay the second maximum, so the utility is $v_i - \max_{j\neq i}\{b_j\} \geq 0$. Since the utility is always non-negative, we have the property.

Part (b)

Find the *maximum* pivot rule $f_i(\mathbf{b}_{-i})$ that ensures individual rationality in the following settings:

i. Bidder i's value v_i is known to satisfy $v_i \in [v_{\min}(i), v_{\max}(i)]$, i.e., the maximum and minimum values of each bidder's valuations is *public* knowledge.

ii. There is a publicly-known constant δ such that any two bidders i and j, we have $|v_i - v_j| \geq \delta$ (i.e., any two bidders' values are at least δ apart).

Solution: **(i)** The function is $\max\{\max_{j\neq i}\{b_j\}, v_{\min}(i)\}$ since each term in the maximum is a lower bound on v_i. The analysis mimics part (a). **(ii)** The function is $\max_{j\neq i}\{b_j\} + \delta$. In case player i wins, if we assume truth-telling, $\max_{j\neq i}\{b_j\}$ is the second maximum valuation. Since we are guaranteed that the highest is at least δ apart, we again get a lower bound on v_i. From here is easy to conclude as in part (a).

Part (c)

Argue that the pivot rule is dominant-strategy incentive compatible for the following choices of f

i. The function from (b) part i.

ii. The function from (b) part ii.

iii. $f_i(\mathbf{b}_{-i}) = \max_{j\neq i}\{b_j\}$

Solution: **(i)** Assume player i losses by betting v_i. Clearly he cannot do better by betting $b_i < v_i$, so let us examine if $b_i > v_i$ yields a non-zero utility. To get the item i needs to bid b_i at least the current maximum, in which case he will have to pay at least $\max_{j\neq i} b_j > v_i$ which yields a negative utility, so he is better of betting v_i. Now assume player i wins by betting v_i, so he gets non-negative utility by part (b). Betting $b_i > v_i$ will not change neither the pay-off nor the allocation, so he has no incentive to do this. Finally, as long as he is the maximum, betting $b_i < v_i$ will not change his utility, but, if he bets lower than the current second maximum, he will get a zero utility. In any case he has no incentive to report different than v_i, so the property holds.

For parts **(ii)** and **(iii)** the analysis is identical. Note that the crucial observation is that, if i gets the item, he does not affect his pay-off by changing his bet (unless he bets so low that he losses the item).

Part (d)

Finally, consider a *procurement auction*, where you want to buy an item (or enter into a contract for some work) from among a group of n sellers. Each seller has a private cost c_i which is known to lie in a (publicly-known) range $[c_{\min}(i), c_{\max}(i)]$. Consider a mechanism that collects bids from the sellers, chooses the seller i^* with the lowest bid, and then pays i^* an amount $p_{i^*}(b_{i^*}, \mathbf{b}_{-i})$. What is the minimum payment that you can offer such that the mechanism is DSIC and IR.

Solution: Note that this is analogous as part **(b.i)**. By identifying payments as negative prices and costs as negative valuations, we get that the payment is zero if i is not the minimum cost and $\min\{\min_{j\neq i}\{b_j\}, c_{\max}(i)\}$ in case i is the minimum cost.

Problem 3: Welfare maximization and externality pricing

Part (a)

Consider an arbitrary single-parameter environment, with feasible set $\mathcal{X}$. Given values v_i, the welfare-maximizing allocation rule is $\mathbf{x}(\mathbf{v}) = \arg\max_{(x_1,\dots,x_n)\in\mathcal{X}} \sum_{i=1}^n v_i x_i$. Prove that this allocation rule is monotone. You can assume for convenience that all values are distinct (or more generally, that ties are broken in some deterministic and consistent way, for example, lexicographically.)

Solution: Assume that i bets v_i and gets the item and let x^* be the allocation. Denote the welfare of an allocation as $w(x,v) = \sum_{i=1}^n v_i x_i$. It must be that

$$w(x^*, v) = \sum_{i=1}^n v_i x_i^* \geq \max_{x\in\mathcal{X}, x_i < x_i^*} \sum_{i=1}^n v_i x_i.$$

Proceed by contradiction and assume that i now bets $b_i > v_i$ and the optimal allocation is $\bar{x}$ with $\bar{x}_i < x_i^*$. By our first inequality, $w(\bar{x}, b_i, v_{-i}) = (b_i - v_i)\bar{x}_i + \sum_j v_j \bar{x}_j \leq (b_i - v_i)\bar{x}_i + w(x^*, v) < w(x^*, b_i, v_{-i})$, which is a contradiction with the fact that $\bar{x}$ maximizes welfare.

Part (b)

Next, consider feasible sets $\mathcal{X}$ that contain only $0-1$ vectors, i.e., each bidder either wins or loses. Now, given any monotone allocation rule $\mathbf{x}(\mathbf{b})$, for any bidder i and other bids $\mathbf{b}_{-i}$, argue that the Myerson payment rule can be written as:

$$p(b_i, \mathbf{b}_{-i}) = \begin{cases} 0 & \text{if } x_i(b_i, \mathbf{b}_{-i}) = 0 \\ b_i^*(\mathbf{b}_{-i}) & \text{if } x_i(b_i, \mathbf{b}_{-i}) = 1 \end{cases}$$

where $b_i^*(\mathbf{b}_{-i})$ is bidder i's *critical bid*, i.e., the lowest bid at which i gets a non-0 allocation.

Solution: We know from class that we can write $p_i(b_i, b_{-i}) = b_i x_i(b_i) - \int_0^{b_i} x_i(z)dz$. By definition of critical bid, this is the same as

$$p_i(b_i, b_{-i}) = b_i x_i(b_i) - \mathbb{1}_{\{b_i > b_i^*\}} \int_{b_i^*}^{b_i} x_i(z)dz = b_i x_i(b_i) - \mathbb{1}_{\{b_i > b_i^*\}}(b_i - b_i^*).$$

Part (c)

For feasible sets $\mathcal{X}$ containing only $0-1$ vectors, we can identify each feasible allocation with a 'winning set' of bidders. Assume further that the environment is 'downward closed', meaning that subsets of a feasible set are again feasible. Prove that, when $\mathcal{S}^*$ is the set of winning bidders and $i \in \mathcal{S}^*$, then is critical bid equals the difference between (i) the maximum surplus of a feasible set that excludes i (you should assume there is at least one such set); and (ii) the surplus $\sum_{j \in \mathcal{S}^* \setminus \{i\}} v_j$ of the bidders other than i in the chosen outcome $\mathcal{S}^*$. Also, is this difference always nonnegative?

Solution: The critical bid is the minimum b_i such that

$$\max_{S \in \mathcal{X}, i \in S} \sum_{j \in S} b_j > \max_{S \in \mathcal{X}, i \notin S} \sum_{j \in S} b_j \iff b_i + \sum_{j \in S^*, j \neq i} b_j > \max_{S \in \mathcal{X}, i \notin S} \sum_{j \in S} b_j.$$

Rearranging terms we get the result. To see that this is non-negative, note that the downward closed property ensures $S^* \setminus \{i\} \in \mathcal{X}$.

Part (d)

To see how to use the above result, consider the knapsack auction we discussed in class (for allocating TV advertisements to ad slots). We want to choose ads to fill a slot of length at most 120 seconds. Bidders have private values v_i and public ad-lengths ℓ_i. In particular, consider a setting with 4 bidders $\{a, b, c, d\}$, with private values (v_a, v_b, v_c, v_d) and lengths $(60s, 40s, 40s, 40s)$. Now suppose all bidders truthfully report their bids, and the auctioneer finds that the optimal allocation is to choose ads b, c and d (note that this is feasible). Use part c to compute the Myerson payments for all the bidders.

Solution: The payment of a is zero. For bidder b the payment is

$$\max\{v_a + v_c, v_a + v_d, v_c + v_d\} - (v_c + v_d).$$

For c and d the expression is symmetric.

Problem 4: The Myerson optimal-revenue auction

In this question, we will get some practice with the Myerson optimal-revenue auction.

Part (a)

Consider an auction with k identical goods, with at most one given to each bidder. There are n bidders whose valuations are i.i.d. draws from a regular distribution F. Describe the optimal auction in this case.

Solution: For the optimal DSIC auction for any general setting, we perform the following steps:

1. Ask each bidder i for their value v_i
2. Compute the *virtual value* $\phi_i(v_i) = v_i - (1 - F_i(v_i))/f_i(v_i)$
3. Find the allocation $\underline{\mathbf{x}}^* \in \mathcal{X}$ that maximizes the virtual welfare $\sum_{i=1}^n \phi_i(v_i)x_i^*$
4. Charge each bidder the Myerson price. In particular, for $x_i \in \{0, 1\}$, we know from above that for welfare maximization, for any bidder i with $x_i^* = 1$, we charge a price p_i equal to the critical bid (see question 3b). In this case, we now need to charge p_i such that $\phi_i(p_i)$ corresponds to the critical virtual bid, i.e., the lowest virtual bid at which i gets a non-0 allocation.

Now for k identical goods, recall we showed in class that the welfare-maximizing auction corresponded to picking the bidders with the k highest bids, and charging them the $(k+1)^{st}$ highest bid. Let the bids be sorted as $b^{(1)} > b^{(2)} > \ldots > b^{(n)}$; since F is regular, we know that the virtual values also have the same order, i.e., $\phi(b^{(1)}) > \phi(b^{(2)}) > \ldots > \phi(b^{(n)})$. To maximize virtual welfare, we consider two cases:

1. If the virtual value of the k^{th} highest bidder is non-negative (i.e., $\phi(b^{(k)}) \geq 0$), then we allocate an item to each of the top k bidders, and charge them all p such that $\phi(p) = \max\{\phi(b^{(k+1)}), 0\}$, i.e., $p = \max\{b^{(k+1)}, \phi^{-1}(0)\}$.
2. If $\phi(b^{(k)}) < 0$, then we find the largest $k' < k$ such that $\phi(b^{(k')}) \geq 0$, allocate an item to each of the top k' bidders, and charge them all $p = \phi^{-1}(0)$.

Note that the above auction is identical to setting a reserve price of $\phi^{-1}(0)$ for each item, and then offering an item to each of the top k bidders at a price equal to the $k+1^{th}$ highest bid or the reserve price, whichever is higher.

Part (b)

Next, consider a single-item auction with independent but non-identical values; in particular, assume bidder is valuation is drawn from its own regular distribution F_i.

i. Give a formula for the winners payment in the optimal revenue auction, in terms of the bidders virtual valuation functions.

ii. Show by example that, in an optimal auction, the highest bidder need not win, even if he has a positive virtual valuation.

Hint: For the last part, a simple setting with two bidders with valuations from different uniform distributions suffices.

Solution: For part i, we first collect bids, and compute virtual bids $\phi_i(b_i)$ for each bidder i. Let i and j be the top virtual bids, i.e., $\phi_i(b_i) > \phi_2(b_j)$. We then offer the item to bidder i, the highest virtual bidder at a price $p_i = \max\{\phi_i^{-1}(0), \phi_i^{-1}(\phi_j(b_j))\}$.

For part ii, consider two bidders i and j, where $F_i \sim$ Uniform$[0,5]$ and the other has $F_j \sim$ Uniform$[0,10]$. Note that if $F \sim$ Uniform$[0,M]$, then $\phi(v) = v - (1 - v/M)/(1/M) = 2v - M$. Thus $\phi_i(b_i) = 2b_i - 5$ and $\phi_i(b_j) = 2b_j - 10$. Now suppose $b_i = 4$ and $b_j = 6$ – then although $b_i < b_j$, we have $\phi_i(4) = 2 \times 4 - 5 = 3$ and $\phi_i(6) = 2 \times 6 - 10 = 2$; thus, we allocate to the second highest bidder in this case.